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MENSURATION

OF

LINES, SURFACES, AND VOLUMES.

BY

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PREFACE.

THE following pages have been transcribed from notes made by the author from time to time in teaching Mensuration in his own classes, and are intended for those pupils only who have previously acquired some knowledge of Geometry and Algebra. Indeed it is not until a pupil has acquired this knowledge that he can take up the subject with any degree of intelligence, or derive any educational advantage from its study.

The subject, however, is so important, and its results are of so much practical utility, that it is not unusual to supplement the ordinary course of arithmetic by the addition of rules and exercises on Mensuration. These rules the arithmetical pupil may follow, and the exercises he may work; but beyond taxing his memory and exercising his fingers, they can be of little or no advantage to him. On the other hand, when he enters upon the study of Mensuration with a knowledge of Geometry and Algebra, he can frame and investigate rules for himself, and thus render the subject an efficient means

of mental discipline. Besides, the results of Mensuration, so important in themselves, afford the student a ready means of illustrating and interpreting geometrical processes and algebraical formulæ.

The author has to express his obligations to Mr J. J. A. BLACK, his able assistant, who has carefully wrought out the exercises proposed, and added his results in the form of answers at the end of the book.

ROYAL HIGH SCHOOL,
EDINBURGH, *March* 1873.

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MENSURATION

OF

LINES, SURFACES, AND VOLUMES.

MENSURATION is that branch of mathematical science which treats of the *measurement* of *lines*, *surfaces*, and *volumes*.

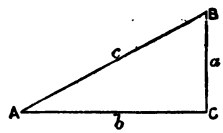
To *measure* a magnitude of any kind, is to find how often it contains another magnitude of the same kind called the *unit;* thus, to measure a *surface*, is to find how often it contains the unit of surface; similarly, to measure a *volume*, is to find how often it contains the unit of volume. In measuring such magnitudes, however, we do not do so by directly applying to the magnitude considered the unit magnitude; geometry affords us the means of deducing their measures, from the length of one or more lines connected with them.

The number which expresses how often a magnitude contains the unit, is called the *numerical measure* of that magnitude.

I.—MENSURATION OF LINES.

THE TRIANGLE.

PROPOSITION I.—*Having given any two sides of a right-angled triangle, to find the remaining side.*



(1.) In the right-angled triangle ABC, having given the sides AC, CB, we find the hypothenuse AB; thus (*Euclid*, I. 47):

$$AB^2 = AC^2 + CB^2;$$

$$\therefore AB = \sqrt{AC^2 + CB^2} \quad \ldots \quad (1).$$

Interpreting this formula, we see that the hypothenuse is found *by taking the square root of the sum of the squares* of the sides.

(2.) Having given the hypothenuse AB, and one of the other sides as AC, we find the remaining side CB; thus:

$$CB^2 = AB^2 - AC^2;$$

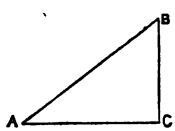
$$\therefore CB = \sqrt{AB^2 - AC^2} \quad \ldots \quad \ldots \quad (2)$$

$$= \sqrt{(AB + AC)(AB - AC)} \quad (3).$$

Interpreting formula (2), we find the remaining side, *by taking the square root of the difference of the squares of the hypothenuse and given side.*

Interpreting formula (3), we find the remaining side, *by taking the square root of the product of the sum and difference of the given sides.*

Numerical Examples.



1. In a right-angled triangle. Given AC = 400, BC = 300; find AB.

Here $AB = \sqrt{(400)^2 + (300)^2}$

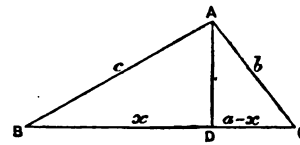
$= \sqrt{250000} = 500.$

2. Given AB = 60, BC = 25; find AC.

Here $AC = \sqrt{(60)^2 - (25)^2} = \sqrt{2975} = 54{\cdot}543.$

PROP. II.—*Given the three sides of a triangle, to find the length of the perpendicular upon any of the sides from the opposite angle.*

In the triangle ABC, let $BC = a$, $AC = b$, $AB = c$; find the length AD of the perpendicular from A on BC.



If one of the segments of the base is known, such as BD, we can at once find AD, the remaining side of the right-angled triangle ABD. First, to find BD,

Let $BD = x$; $\therefore DC = a - x$.

Now $AD^2 = c^2 - x^2$ in the triangle ABD.

Again, $AD^2 = b^2 - (a-x)^2$ " ACD;

$$\therefore c^2 - x^2 = b^2 - (a-x)^2,$$

$$\text{or } c^2 - x^2 = b^2 - a^2 + 2ax - x^2;$$

$$\therefore x = \frac{a^2 + c^2 - b^2}{2a} = BD;$$

$$\therefore AD^2 = c^2 - x^2 = c^2 - \left(\frac{a^2 + c^2 - b^2}{2a}\right)^2$$

$$= \left(c + \frac{a^2 + c^2 - b^2}{2a}\right)\left(c - \frac{a^2 + c^2 - b^2}{2a}\right)$$

$$= \frac{2ac + a^2 + c^2 - b^2}{2a} \times \frac{2ac - a^2 - c^2 + b^2}{2a}$$

$$= \frac{(a+c)^2 - b^2}{2a} \times \frac{b^2 - (a-c)^2}{2a}$$

$$= \frac{(a+c+b)(a+c-b)(b+a-c)(b-a+c)}{4a^2};$$

$$\therefore AD = \sqrt{\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4a^2}};$$

$$\therefore AD = \frac{1}{2a}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}.$$

In order to simplify this expression, let s denote half the sum of the sides.

That is, let $a+b+c=2s$;

$$\therefore b+c-a=2(s-a)$$
$$c+a-b=2(s-b)$$
$$a+b-c=2(s-c)$$

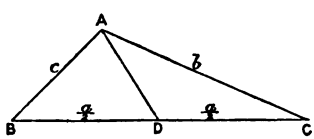
$$\therefore \text{AD}=\frac{1}{2a}\sqrt{2s\cdot 2(s-a)2(s-b)2(s-c)}$$
$$=\frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)} \quad \ldots \quad \ldots \quad \ldots \quad (1).$$

From the form assumed here for the perpendicular on side a, we can at once write down the perpendiculars on sides b and c, which are respectively

$$\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)},$$
$$\text{and } \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

Interpreting formula (1), we have the following rule for finding the perpendicular on any side: *Add the three sides together, and from half the sum subtract each side separately, multiply half the sum and the three remainders, and divide twice the square root of this product by the side on which the perpendicular is drawn.*

PROP. III.—*Given the three sides of a triangle, find the length of the line drawn from one of the angles to the bisection of the opposite side.*



Let a, b, c denote the lengths of the sides opposite angles A, B, C respectively; it is required to find the length of AD drawn from A to the middle of BC.

(*Euclid*, II. A) $\text{AB}^2+\text{AC}^2=2\text{BD}^2+2\text{AD}^2$,

or $c^2+b^2=\frac{a^2}{2}+2\text{AD}^2$;

$$\therefore AD^2 = \frac{b^2 + c^2 - \frac{a^2}{2}}{2} = \frac{2b^2 + 2c^2 - a^2}{4};$$

$$\therefore AD = \tfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \quad \ldots \quad \ldots \quad (1).$$

Similarly, for the line drawn from B to bisection of side b, we have

$$\tfrac{1}{2}\sqrt{2a^2 + 2c^2 - b^2};$$

and for the line from C to bisection of side c,

$$\tfrac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}.$$

Cor.—Since $b + c > a$;

$\therefore b^2 + c^2 + 2bc > a^2$;

or $b^2 + c^2 > a^2 - 2bc$;

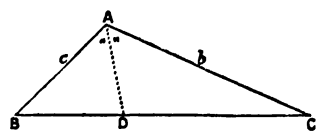
but $b^2 + c^2 > 2bc$;

$\therefore 2b^2 + 2c^2 > a^2$;

so that the right-hand member of (1) can never be negative, and consequently the root is always real.

PROP. IV.—*Given the three sides of a triangle, find the length of the line bisecting any of the angles.*

Let, as before, a, b, c denote the sides opposite angles A, B, and C. Bisect angle A by line AD, meeting BC in D. It is required to find the length of AD.



We have (*Euclid*, VI. 3)

$$c : b :: BD : DC;$$

$$\therefore c + b : b :: BD + DC : DC.$$

$$\text{Or } c + b : b :: a : DC;$$

$$\therefore DC = \frac{ab}{b + c};$$

$$\text{Similarly } DB = \frac{ac}{b + c}.$$

Having thus found the segments of the base, we can at once find AD, for (*Euclid*, VI. B)

We have $BD \cdot DC + AD^2 = AB \cdot AC$;

$$\therefore AD^2 = AB \cdot AC - BD \cdot DC$$

$$= bc - \frac{ab}{b+c} \times \frac{ac}{b+c}$$

$$= bc - \frac{a^2}{(b+c)^2} bc = bc \left\{ 1 - \frac{a^2}{(b+c)^2} \right\}$$

$$= \frac{bc}{(b+c)^2} \left\{ (b+c)^2 - a^2 \right\} = \frac{bc}{(b+c)^2} \left\{ (b+c+a)(b+c-a) \right\}$$

$$\therefore AD = \frac{1}{b+c} \sqrt{bc(a+b+c)(b+c-a)}.$$

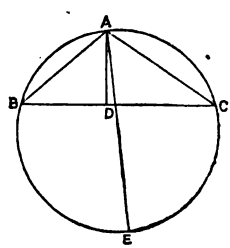
Similarly, for bisectors meeting sides b and c, we have respectively the expressions

$$\frac{1}{a+c} \sqrt{ac(a+b+c)(a+c-b)},$$

and $$\frac{1}{a+b} \sqrt{ab(a+b-c)(a+b-c)}.$$

Cor.—It will be observed that when $c = b$, the results of Propositions II. III. and IV. are alike; and this was to be expected, for when the triangle is isosceles, the perpendicular on the base, the line drawn to the middle of the base, and the line bisecting the vertical angle, all coincide.

PROP. V.—*Given the three sides of a triangle, find the diameter of the circumscribing circle.*



Let a, b, c, as before, denote sides opposite angles A, B, and C. Let p denote the perpendicular from angle A on BC, and D the diameter of the circumscribing circle.

We have (*Euclid*, VI. c),

$$Dp = bc;$$

$$\therefore D = \frac{bc}{p}.$$

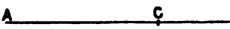
But p, the perpendicular, can be found (Prop. II.), and is equal to $\frac{2}{a} \sqrt{s(s-a)(s-b)(s-c)}$.

$$\therefore D = \frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}}.$$

Remark.—It may here be observed that the above expression for the diameter of the circumscribing circle is symmetrical with respect to the sides, as might have been expected.

PROP. VI.—*To divide a line into extreme and mean ratio.*

Let $AB = a$ be the given line, and let C be the point of division. Then $AB \cdot BC = AC^2$.



It is required to find AC. Let $AC = x$ = the greater of the two segments.

$$\therefore a(a-x) = x^2, \text{ or } a^2 - ax = x^2;$$

$$\therefore x^2 + ax = a^2$$

$$x = \frac{-a \pm \sqrt{a^2 + 4a^2}}{2}$$

$$= -\frac{a}{2} \pm \frac{a}{2}\sqrt{5} = \frac{a}{2}\left\{\sqrt{5} - 1\right\}.$$

Exercises (1).

1. Find the hypothenuse of a right-angled triangle of which the base and perpendicular are:

(1.) 15 and 20.
(2.) 12 and 15.
(3.) 25 and 30.
(4.) 17 and 17.
(5.) 625 and 515.
(6.) 21·4 and 22.

2. Find the base of a right-angled triangle of which the hypothenuse and perpendicular are:

(1.) 200 and 112.
(2.) 12·5 and 6·25.
(3.) 17·2 and 16·1.

3. Find the perpendicular of a right-angled triangle of which the hypothenuse and base are:

(1.) 100 and 50.
(2.) 1·25 and ·05.
(3.) 500 and 300.

4. Find the diagonal of a rectangle whose adjacent sides are 15 and 12.

5. Find the perimeter of a right-angled triangle of which the base is 36 and the hypothenuse is 60.

6. The diagonal of a rectangle is 100, and one of the sides 50; find the other side.

7. The side of a square is 5 feet; find its diagonal.

8. The diagonal of a square is 100; find its side.

9. Find the perimeter of a square whose diagonal is 12·5 feet.

10. Find the perimeter of a square inscribed in a circle whose diameter is 1.

11. Find the radius of the circle circumscribing a square whose side is 6 feet.

12. Find the diagonal of a cube, the edge of which is 12 feet.

13. The diagonal of a cube is 15 feet; find its edge.

14. Find the distance from the centre of a circle of a chord whose length is 30 feet in a circle whose radius is 20.

15. The radius of a circle is 15 inches, the perpendicular from the centre on a chord is 12 inches; find the length of the chord.

16. Find the length of the *least* chord, drawn through a point 3 inches from the centre of a circle whose diameter is 10 inches.

17. A ladder 50 feet long reaches a point in a wall 30 feet high; how far is the foot of the ladder from the bottom of the wall?

18. A ladder 50 feet long reaches a point in a wall 30 feet high; the foot of the ladder is drawn out along the ground 3 feet; how far will the top of the ladder slip down the wall?

19. A pole 60 feet high is broken by the wind, and the top strikes the ground 10 feet from the foot of the pole; how far from the ground is the pole broken?

20. In an equilateral triangle, whose side is 10 feet, find the length of the perpendicular from any of the angles on the opposite side.

21. The perpendicular height of an equilateral triangle is p; find the length of its side.

22. The difference between the diagonal and side of a square is 5 feet; find the length of the side.

23. The sum of the diagonal and side of a square is 5 feet; find the side.

24. Find the perimeter of an isosceles triangle, each of whose equal sides is s, and perpendicular on base p.

25. Find the lengths of the perpendiculars on the *longest* and *shortest* sides in the triangles whose sides are:

(1.) 200, 250, 300.	(4.) 3, 4, 5.
(2.) 10, 12, 20.	(5.) 6, 8, 10.
(3.) 17, 18, 19.	(6.) 5, 10, 10.

26. Shew how to find the segments of the sides in each of the above cases.

27. Find also the diameters of the circumscribing circles.

28. Find the lengths of the lines drawn to the middle of the greatest and least sides from the opposite angles in the triangles whose sides are:

(1.) 200, 300, and 350. (2.) 500, 600, and 750.

29. Find also the lengths of the lines bisecting the angle between the *greatest* and *least* side, in each of the above triangles.

30. The base of a triangle is 48 feet, the height 20 feet, and one of the sides 24 feet; find the other side.

31. In a right-angled triangle, having given the sides containing the right angle equal to a and b, find

(1.) The perpendicular from the right angle on the hypothenuse.

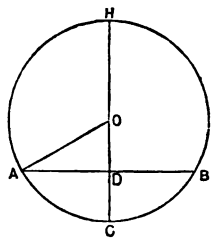
(2.) The segments into which the perpendicular divides the hypothenuse.

32. In a triangle ABC, if Aa be drawn from angle A to the middle of the opposite side, and Bb from B to the middle of the opposite side, intersecting each other in O; prove

$$aO = \tfrac{1}{3}aA.$$

THE CIRCLE.

PROP. VII.—*Having given the height of an arc, and the radius of the circle, to find the length of the chord.*



Let DC, the height of the arc ACB, $= h$, and radius of circle $= r$; to find the length h of the chord AB. In the right-angled triangle AOD, the hypothenuse $AO = r$, the perpendicular $OD = r - h$;

$$\therefore AD^2 = r^2 - (r-h)^2 = 2rh - h^2;$$

$$\therefore AD = \sqrt{2rh - h^2};$$

$$\therefore AB = l = 2\sqrt{2rh - h^2} \quad \ldots \quad \ldots \quad (1).$$

We might also have obtained AB thus:

$$AD^2 = HD \cdot DC$$
$$= (2r - h)h = 2rh - h^2,$$

or
$$AD = \sqrt{2rh - h^2};$$

$$\therefore l = 2\sqrt{2rh - h^2}, \text{ as before.}$$

Interpreting this formula, we have the following rule for determining the chord: *Multiply twice the radius by the height; from the product subtract the square of the height, and take twice the square root of the remainder.*

Remark.—It may here be noticed that, in the above formula,

when $h = o$, or $2r$, the chord is zero;

and when $h = r$, the chord becomes the diameter.

Cor.—The above formula involves the three quantities, l, r, and h, and expresses the value of l in terms of r and h; but it also enables us to express any one of the three quantities in terms of the other two.

(1.) To find r,

$$\frac{l^2}{4} = 2rh - h^2,$$

$$2rh = \frac{l^2}{4} + h^2 = \frac{l^2 + 4h^2}{4};$$

$$\therefore r = \frac{l^2 + 4h^2}{8h} \quad \ldots \quad \ldots \quad (2).$$

Interpreting this, we have the following rule for finding the radius: *Add the square of the chord to four times the square of the height, and divide the sum by eight times the height.*

(2.) To find h, we have

$$4h^2 - 8hr + l^2 = o,$$

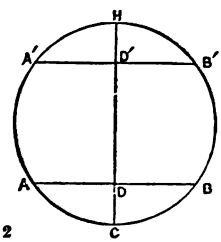
or

$$h^2 - 2r \cdot h + \left(\frac{l}{2}\right)^2 = o \quad \ldots \quad (3).$$

Solving for h, we have

$$h = r \pm \tfrac{1}{2}\sqrt{4r^2 - l^2} \quad \ldots \quad (4).$$

The *two* roots of this equation shew that we have *two* values of h for each value of l. This will be readily understood by referring to the geometrical figure; for if we inscribe a chord AB equal to l, it is evident that we have another chord A′B′ also equal to l, parallel to AB, and equally distant from the centre; the two heights which correspond to a chord equal to l are CD and CD′. We see also that



$$CD + CD' = 2r,$$

and

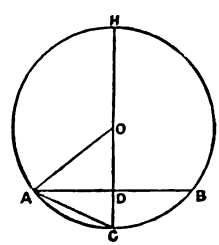
$$CD \cdot CD' = \left(\frac{l}{2}\right)^2.$$

PROP. VIII.—*Having given the side of a regular polygon inscribed in a given circle, to find the side of the regular polygon inscribed with double the number of sides.*

Let AB = l, the side of the given inscribed polygon.

AO = r. CH being the diameter perpendicular to AB through its middle point D, AC will be the side required, and which we will denote by k. We evidently have

$$2r : k :: k : CD;$$



$$\therefore k^2 = 2r \cdot \mathrm{CD} = 2r(r - \mathrm{OD}) = r(2r - 2\mathrm{OD}).$$

But $$\mathrm{OD} = \sqrt{\mathrm{AO}^2 - \mathrm{AD}^2} = \sqrt{r^2 - \left(\frac{l}{2}\right)^2} = \tfrac{1}{2}\sqrt{4r^2 - l^2};$$

$$\therefore k^2 = r\left(2r - \sqrt{4r^2 - l^2}\right);$$

$$\therefore k = \sqrt{r\left(2r - \sqrt{4r^2 - l^2}\right)} \quad \ldots \quad (1).$$

As a particular case, we may take radius equal to unity;

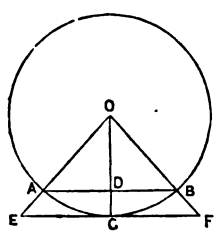
$$\therefore k = \sqrt{2 - \sqrt{4 - l^2}} \quad \ldots \quad \ldots \quad (2).$$

Cor.—Equation (1) expresses k in terms of l, but it also gives us l in terms of k; for, solving, we have

$$l = \frac{k}{r}\sqrt{4r^2 - k^2}.$$

In other words, having given the side of a regular polygon of a certain number of sides, we can find the length of a side of the polygon of *half* the number.

PROP. IX.—*Having given the side of a regular polygon inscribed in a given circle, to find the side of the regular polygon of the same number of sides circumscribing the circle.*



Let AB = l, the side of the inscribed polygon; r = radius of the circle. Draw the tangent EF through the middle point C of the arc AB, meeting OA and OB produced in E and F. EF will evidently be the side required, which we shall denote by L.

In the similar triangles AOD, EOC, we have

$$\mathrm{EC} : \mathrm{AD} :: \mathrm{OC} : \mathrm{OD},$$

or $$\frac{\mathrm{L}}{l} = \frac{r}{\mathrm{OD}}.$$

But $$\mathrm{OD} = \sqrt{r^2 - \left(\frac{l}{2}\right)^2} = \tfrac{1}{2}\sqrt{4r^2 - l^2};$$

$$\therefore \frac{L}{l} = \frac{2r}{\sqrt{4r^2 - l^2}};$$

$$\therefore L = \frac{2rl}{\sqrt{4r^2 - l^2}} \quad \ldots \quad (1),$$

and
$$l = \frac{2rL}{\sqrt{4r^2 + L^2}}.$$

As a particular case, we may take radius equal to unity;

$$\therefore L = \frac{2l}{\sqrt{4 - l^2}}.$$

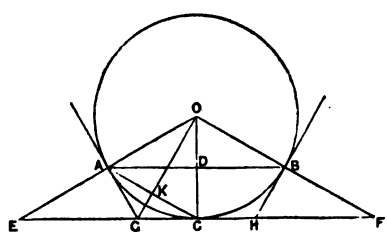
Cor.—Knowing the side of a regular polygon of n sides inscribed in a circle, we can, by the repeated application of the preceding proposition, find the side, and consequently the perimeter of a regular polygon of

$$2n,\ 4n,\ 8n,\ 16n,\ 32n,\ \text{\&c. sides,}$$

inscribed within or described about the given circle.

PROP. X.—*Given the perimeter of a regular polygon inscribed within a circle, and the perimeter of a similar circumscribed polygon, to find the perimeters of the regular inscribed and circumscribed polygons of double the number of sides.*

Let AB = side of inscribed polygon; EF, the tangent through the middle point, C, of arc AB, meeting OA and OB produced in E and F; then will EF = side of circumscribed figure. Join AC. AC is the side of figure inscribed with double the number of sides. Through A and B draw tangents AG,



BH. GH is the side of the figure circumscribed, with double the number of sides.

Let p and P denote respectively the perimeters of the inscribed and circumscribed figures; and p' and P′ the perimeters respectively of the inscribed and circumscribed figures of double the number of sides.

Since OE is the radius of the circle circumscribing the polygon whose perimeter is P, and OC that of the circle circumscribing the similar polygon whose perimeter is p, we have

$$\frac{P}{p}=\frac{OE}{OC}.$$

But since OG bisects the angle EOC, we have (*Euclid*, VI. 3):

$$\frac{OE}{OC}=\frac{EG}{GC};$$

$$\therefore \frac{P}{p}=\frac{EG}{GC};$$

$$\therefore \frac{P+p}{2p}=\frac{EG+GC}{2GC}=\frac{EC}{GH}.$$

Now, GH is a side of the polygon whose perimeter is P′, and is contained in P′ the same number of times that EC is contained in P, or

$$\frac{EC}{GH}=\frac{P}{P'};$$

$$\therefore \frac{P+p}{2p}=\frac{P}{P'};$$

$$\therefore P'=\frac{2pP}{P+p} \quad \ldots\ldots \quad (1).$$

Again, the triangles ACD and GKC are similar;

$$\therefore \frac{AD}{AC}=\frac{KC}{CG}.$$

But since AD is contained in P the same number of times that AC is contained in p',

$$\frac{AD}{AC}=\frac{p}{p'};$$

and similarly,

$$\frac{KC}{CG}=\frac{p'}{P'};$$

$$\therefore \frac{p}{p'}=\frac{p'}{P'};\quad \therefore p'=\sqrt{pP'};$$

or
$$p'=p\sqrt{\frac{2P}{P+p}} \quad \ldots \quad \ldots \quad (2).$$

PROP. XI.—*To find the side of a regular decagon inscribed in a circle.*

In *Euclid* (IV. 10), it can be shewn that the angle BAD at the centre of the circle is an angle of *one-fifth* of two right angles, or *one-tenth* of four; *ten* such angles can therefore be made round the centre, consequently the line BD is the side of a regular decagon inscribed in the circle.

But BD = AC, the greater of the two segments, which is obtained by dividing AB into extreme and mean ratio;

$\therefore$ side of regular decagon $=\frac{R}{2}(\sqrt{5}-1)$, Prop. VI.

Exercises (2).

1. What must be the radius of a circle in which a chord of 15 inches is 5 inches from the centre?

2. In a circle whose diameter is 20 feet, a chord of 12 feet is inscribed; find the heights of the arcs.

3. The height of an arc is 12 inches, and the chord of half the arc is 36; find the diameter of the circle.

4. The height of an arc is 20 inches, and the diameter of the circle is 60 inches; find the chord of half the arc.

5. In a circle whose diameter is 5 feet 4 inches, the chord of the arc is 20 inches; find the chord of half the arc.

6. In a circle, the chord of half the arc is twice the height; prove that the height is half the radius.

7. The chord of an arc is 20 feet, and the chord of half the arc is 30; find the diameter.

8. In a circle whose radius is R, find

(1.) Side of inscribed regular hexagon.
(2.) Side of circumscribed regular hexagon.
(3.) Side of inscribed equilateral triangle.
(4.) Side of circumscribed equilateral triangle.

9. In a circle whose radius is R, shew that the side of the regular inscribed octagon

$$= R\sqrt{2 - \sqrt{2}}.$$

10. In a circle whose radius is R, shew that the side of the regular inscribed dodecagon

$$= R\sqrt{2 - \sqrt{3}}.$$

11. Compare the perimeters of the inscribed equilateral triangle, the square and the octagon.

12. Find the radius of a circle in which two parallel chords of 6 and 8 inches are 1 inch apart.

13. What is the length of the tangent drawn from a point 10 inches from the centre of a circle whose radius is 8 inches?

14. How far from the centre of a given circle must a point be taken, through which if a tangent be drawn, the tangent will be equal to

(1.) The radius of the circle?
(2.) The diameter of the circle?

15. The distance between the centres of two circles, whose radii are R and r, is D; find the lengths of their common tangents. What do these lengths become when

$$D = R + r?$$

16. Find the side of a regular decagon inscribed in a circle whose radius is R.

17. If d = the diagonal of a regular pentagon inscribed in a circle whose radius is R, prove that

$$d = \frac{R}{2}\sqrt{10 + 2\sqrt{5}}.$$

PROP. XII.—*To find an approximate expression for the ratio of the circumference of a circle to its diameter.*

We shall take the diameter of the circle as given, and determine the perimeters of an inscribed and similarly circumscribed polygon. We shall then determine the perimeters of inscribed and circumscribed polygons of double the number of sides. Taking the last found perimeters as given, we will then determine the perimeters of polygons of double the number of sides by the same method, and so on. As the number of sides increases, it may be shewn that the perimeters approach nearer and nearer to the circumference; hence their successively determined values will be nearer and nearer approximations to the value of the circumference.

Taking the diameter of the circle as 1, we will begin by inscribing and circumscribing a square, and finding their perimeters.

Here we have $p = 2\sqrt{2} = 2{\cdot}8284271$,

and $P = 4$.

Having found P and p, we find by Proposition X. P' and p';

thus $$P' = \frac{2p + P}{P + p} = 3{\cdot}3137085,$$

and $$p' = \sqrt{p \times P'} = 3{\cdot}0614675.$$

Then taking P' and p' as given quantities, we put

$$P = 3{\cdot}3137085,$$

and $$p = 3{\cdot}0614675;$$

and find, by the same formulæ, for the polygon of 16 sides,

$$P' = 3{\cdot}1825979,$$

$$p' = 3{\cdot}1214452.$$

Continuing this process, the results will be found as in the following table:

Number of Sides.	Perimeter of Circumscribed Polygon.	Perimeter of Inscribed Polygon.
4	4·0000000	2·8284271
8	3·3137085	3·0614675
16	3·1825979	3·1214452
32	3·1517249	3·1365485
64	3·1441184	3·1403312
128	3·1422236	3·1412773
256	3·1417504	3·1415138
512	3·1416321	3·1415729
1024	3·1416025	3·1415877
2048	3·1415951	3·1415914
4096	3·1415933	3·1415923
8192	3·1415928	3·1415926

From the last two numbers of this table, we learn that the circumference of a circle whose diameter is unity is *less* than

$$3{\cdot}1415928,$$

and *greater* than

$$3{\cdot}1415926.$$

It follows therefore that

$$3{\cdot}1415927$$

represents the circumference to within a unit of the seventh decimal place.

It is usual to represent this number by the symbol

$$\pi,$$

$$\therefore \text{ circumference} = \pi \times \text{diameter}.$$

If we convert the expression for π into a series of continued fractions,* we will find for it

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \text{ \&c.}$$

It is usual, in practice, to take for π, $\frac{22}{7}$, or $3\frac{1}{7}$, as this is sufficiently near for all practical purposes.

Let C denote the circumference,

D the diameter,

r the radius.

We then have the following formulæ:

* See author's treatise on the *Theory of Arithmetic.*

$$C = \pi D; \quad \therefore D = \frac{C}{\pi}.$$

$$C = 2\pi r; \quad \therefore r = \frac{C}{2\pi} = \frac{D}{2}.$$

Cor.—Hence the ratio of two circumferences is the same as the ratio of their radii.

Let C and r be the circumference and radius of one circle, and C′ and r' the circumference and radius of another.

Then $C = 2\pi r$,

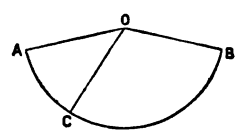
and $C' = 2\pi r'$;

$$\therefore \frac{C}{C'} = \frac{r}{r'}.$$

THE MEASURE OF ANGLES.

PROP. XIII.—*The numerical measure of an angle at the centre of a circle is the same as the numerical measure of its intercepted arc, if the adopted unit of angle is the angle at the centre which intercepts the adopted unit of arc.*

Let AOB be an angle at the centre O, and AB its intercepted arc. Let AOC be the angle which is adopted as the *unit of angle*, and let its intercepted arc AC be the arc which is adopted as the *unit of arc.* Then (*Euclid*, VI. 33),



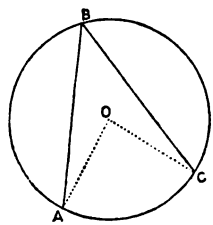
$$\frac{\text{angle AOB}}{\text{angle AOC}} = \frac{\text{arc AB}}{\text{arc AC}}.$$

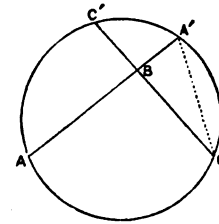
But the first of the above ratios is the measure of the angle AOB, referred to the unit angle AOC; and the second ratio is the measure of the arc AB, referred to the unit arc AC. Therefore, with the adopted units, the numerical measure of the angle AOB is the same as that of the arc AB.

Remark 1.—This proposition, being of frequent application, is usually more briefly though inaccurately

expressed, by saying that *an angle at the centre is measured by the arc on which it stands.* In this statement of the proposition, the condition that the adopted units of angle and arc correspond to each other is understood, and the expression 'is measured by' is used for 'has the same numerical measure as.'

Cor. 1.—An angle, ABC, at the circumference of a circle is measured by *half* the arc AC on which it stands, since it is equal to half the angle AOC at the centre (*Euclid*, III. 20).





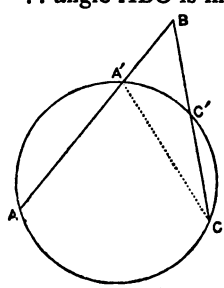
Cor. 2.—An angle, ABC, standing within the circle, but not at the centre, is measured by *half the sum of the arcs* AC *and* A′C′. For joining A′C, we have

angle ABC = angle AA′C + angle C′CA′;

but angle AA′C is measured by half-arc AC,

and angle C′CA′ is measured by half-arc A′C′;

$$\therefore \text{ angle ABC is measured by } \frac{\text{arc AC} + \text{arc A′C′}}{2};$$



or, in other words, the angle ABC is equal to that angle at the centre standing on an arc equal to half the sum of the intercepted arcs AC and A′C′.

Cor. 3.—An angle ABC standing without the circle is measured by *half the difference* of the intercepted arcs AC and A′C′. For joining A′C, we have

angle ABC = angle AA′C − angle C′CA′;

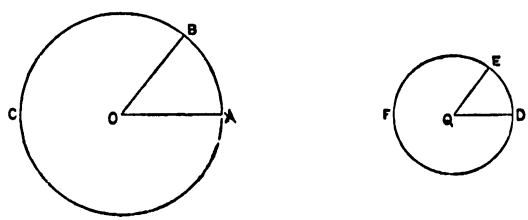
but angle AA′C is measured by half-arc AC,

and angle C′CA′ is measured by half-arc A′C′;

$$\therefore \text{angle ABC is measured by } \frac{\text{arc AC} - \text{arc A′C′}}{2};$$

or, in other words, the angle ABC is equal to that angle at the centre which stands on an arc equal to half the difference of the intercepted arcs AC and A′C′.

Remark 2.—The measurement of angles, then, is reduced to the measurement of arcs, and the arc which is taken for the unit is an arc in the circle which is equal to the radius. The unit angle will be an angle at the centre subtended by the unit arc; and we have seen that the ratio of the circumference of a circle to its radius is an invariable number, so that, in a circle of any radius whatever, the angle at the centre, which is subtended by an arc equal to the radius, is always of the same magnitude. That is, in the circles ABC, DEF,



of different radii, AO and DQ, the angles AOB, DQE, subtended at their centres by arcs equal to their radii, are equal, whatever be the magnitudes of the circles.

When an angle is measured by referring it to the unit angle, or, which is the same thing, when its arc is measured by referring it to the unit arc, the measure we obtain is said to be the *circular measure* of the angle.

There is another method adopted for measuring angles, and this we now explain.

The *right angle* is, by its nature, the simplest unit

angle; nevertheless, custom has sanctioned a different unit. The unit of angle generally adopted is an angle equal to $\frac{1}{90}$th part of a right angle, called a *degree*, and denoted by the symbol °. The corresponding unit of arc is $\frac{1}{90}$th part of a quadrant, and is also called a degree. A right angle and a quadrant are both expressed by 90°. Two right angles and a semi-circumference are both expressed by 180°. Four right angles and a whole circumference are both expressed by 360°.

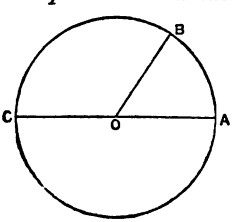
The degree either of angle or arc is subdivided into *minutes* and *seconds*, denoted by the symbols ′ and ″; a minute being the $\frac{1}{60}$th part of a degree, and a second the $\frac{1}{60}$th part of a minute. Fractional parts of a degree less than a second are expressed by decimal parts of a second.

An angle, or an arc of any magnitude, is then numerically expressed by the unit degree and its subdivisions. Thus, for example, an angle equal to $\frac{1}{7}$th of a right angle, as well as its intercepted arc, will be expressed by 12° 51′ 25″·714....

Definition 1.—Two angles are *complemental* when their sum is 90°.

Def. 2.—Two angles are *supplemental* when their sum is 180°.

PROP. XIV.—*To find the number of degrees, minutes, &c. in an angle at the centre of a circle subtended by an arc equal to the radius.*



Let ABC be a circle whose radius is AO. Take the arc AB = to the radius, and join BO. It is required to find the number of degrees, minutes, &c. in the angle AOB.

We have $$\frac{\text{angle AOB}}{\text{angle AOC}} = \frac{\text{arc AB}}{\text{arc AC}};$$

or $$\frac{\text{angle AOB}}{180^\circ} = \frac{\text{radius}}{\text{semi-circumference}};$$

$$\therefore \text{angle AOB} = \frac{\text{radius}}{\pi \times \text{radius}} \times 180^\circ;$$

$$\therefore \text{angle AOB} = \frac{1}{\pi} \times 180^\circ = 57^\circ{\cdot}29577$$

$$= 57^\circ\ 17'\ 45''.$$

This is the magnitude of *the angle in any circle which is subtended by an arc equal to the radius*, and as an arc equal to the radius is taken for the unit arc, so this angle is taken for the unit angle, and serves for measuring the magnitude of other angles.

Numerical Examples.

1. What is the magnitude of that angle whose circular measure is $\frac{1}{2}$? Or, in other words, what is the magnitude of that angle which is subtended by an arc equal to $\frac{1}{2}$ of the radius?

It has been shewn above that an angle subtended by an arc equal to the radius is

$$57^\circ\ 17'\ 45'';$$

therefore that subtended by an arc equal to $\frac{1}{2}$ the radius will be

$$28^\circ\ 38'\ 52''.$$

2. What is the circular measure of an angle of 72°?

The circular measure of 180° is π;

$\therefore 180^\circ : 72^\circ :: \pi :$ circular measure of 72°;

$\therefore$ circular measure of $72^\circ = \frac{72}{180}\pi = \frac{2}{5}\pi$.

Remark.—It has been shewn that

$$\pi = 3{\cdot}14159;$$

and it should be borne in mind that π is *always* equal to this number; but sometimes the young student may find π taken to represent 180°, and in that case he may be reminded that π, still retaining its original signification, is understood to mean an angle which is $3{\cdot}14159$ times the unit angle ($57^\circ\ 17'\ 45''$).

Exercises (3).

1. Find the circumference of a circle whose radius is

(1.) 20 inches. | (3.) 450 yards.
(2.) 3 feet 5 inches. |

2. Find the diameter of a circle whose circumference is

(1.) 100 feet. | (4.) 483 feet.
(2.) 1·4 inch. | (5.) 55 feet.
(3.) 12 feet 6 inches. | (6.) $\frac{1}{2}$ inch.

3. Find the circumference of a 4-inch drain pipe.

4. The difference between the circumference and diameter of a circle is

(1.) d. | (2.) 20 feet.

Find the radius of the circle in both cases.

5. The sum of the circumference and diameter of a circle is

(1.) s. | (2.) 20 feet.

Find the radius of the circle in both cases.

6. A circular grass-plot measures 50 feet in diameter; find its circumference.

7. Find the circumference of a circle whose radius is 12 feet 9 inches.

8. How often will a carriage wheel which is 3 feet 6 inches in diameter turn round in going a distance of 1 mile?

9. A wheel which is a foot and a half in diameter revolves 1209 times; through what distance has it passed?

10. What must be the radius of a wheel which revolves 220 times in passing over one half-mile?

11. Find the circumference described by the point of the minute-hand of a clock $3\frac{5}{8}$ inches in length.

12. Compare the rates at which the points of the hour and minute hands of a watch move, their lengths being respectively $\frac{2}{5}$ and $\frac{7}{8}$ of an inch.

13. Compare the perimeter of a square, and that of the inscribed and circumscribed circles.

14. Compare the circumference of a circle with the perimeters of the inscribed and circumscribed hexagons.

15. The earth moving round the sun at the rate of 1135 miles per minute, takes $365\frac{1}{4}$ days to complete her revolution; shew how to find the distance of the earth from the sun, the orbit being supposed circular.

16. Find the least of the two angles subtended by the hands of a watch at

(1.) 25 minutes past 12 o'clock.
(2.) 35 " 2 "
(3.) 20 " 3 "
(4.) 50 " 5 "
(5.) 5 " 5 "
(6.) 30 " 6 "

17. When between 3 and 4 o'clock will the hands subtend an angle of 45°?

18. In a circle whose radius is 10 inches, find

(1.) The angle subtended at the centre by an arc of 15 inches.
(2.) The angle subtended at the centre by an arc of 20 inches.
(3.) The angle subtended at the centre by an arc of 1 foot.

19. Find also the corresponding angles subtended at the circumference.

20. In a circle whose diameter is 40 feet, find the arc of 25° 30′ at the centre.

21. What must be the radius of a circle, so that an angle of 75° at the centre is subtended by an arc of 36 inches?

22. In a circle whose radius is 1 foot, find the angle subtended by an arc of 1 inch.

23. Find the angle at the circumference which is subtended by an arc of 10 inches in a circle whose radius is 10 inches.

24. Find the number of degrees in an angle formed by two chords intersecting:

(1.) Within the circle, | (2.) Without the circle;

and intercepting arcs equal to m and n, the radius of the circle being r.

25. Find the radius when an angle of 25° at the circumference has an arc of 100 inches.

26. Find the angle whose circular measure is

$$\frac{2}{3},\quad \frac{3}{4},\quad 1,\quad \frac{5}{3},\quad \frac{9}{4},\quad \theta.$$

27. Find the circular measure of the following angles:

20°, 45°, 108°, 235°, 112° 30′, 150°, 300°.

28. Compare the radii of two circles, in which an angle of 40° at the centre of the one and an angle of 60° at the centre of the other have equal arcs.

29. Find the complements of the following angles:

25°, 37° 15′, 42° 25′ 15″.
75°, 81° 10′, 7° 15′.

30. Find the supplements of the following angles:

110° 15′, 71°, 108° 25′.
25° 10′, 60°, 135°.

31. The circular measure of an angle is $\frac{3}{4}$; find

(1.) Circular measure of the complement.
(2.) " " " supplement.

32. What is the magnitude of that angle, the circular measure of whose supplement is three times the circular measure of its complement?

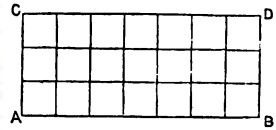
ON OF SURFACES.

SURFACES.

RECTANGLE.

of a rectangle, having given

$= b$, feet, dividing AB into a
into b



gh the
w lines
It is evi-
figure will
number of
which is a
we have a rows of these rectangles,
the whole number of rectangles will be
ab,
area of rectangle $= ab$.
area by S,

$$S = ab \quad \ldots \quad (1).$$

find the area, *we multiply the two adjacent*
From (1) we have

$$a = \frac{S}{b} \quad \ldots \quad (2.)$$

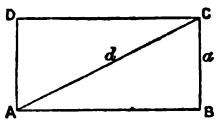
e, having given the area and one side, *to find the*
side, we divide the area by the given side.

n the side of a square equal to a,
idently we have

$$a \cdot a = a^2;$$
$$\therefore a = \sqrt{S}.$$

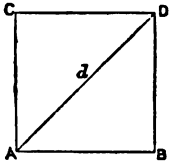
Cor. given the diagonal of a rectangle,
and on find the area.

* We nsidered only the case in which each of the
sides t an exact number of times. For general
proof



$$AB = \sqrt{AC^2 - CB^2}$$
$$= \sqrt{d^2 - a^2};$$
$$\therefore \text{ area} = a\sqrt{d^2 - a^2}.$$

Cor. 3.—Having given the diagonal of a square, to find its area.

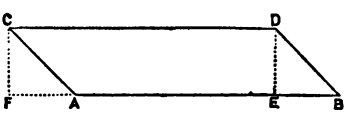


$$AB^2 + BD^2 = d^2,$$
or
$$2AB^2 = d^2;$$
$$\therefore AB^2 = \frac{d^2}{2} = \text{area}.$$

Hence we find the area of a square, when the diagonal is given, *by taking half the square of the diagonal.*

THE PARALLELOGRAM.

PROP. II.—*To find the area of a parallelogram, having given its base and perpendicular height.*



Let $AB = a$, $DE = p$. Draw through C, CF perpendicular to AB, and meeting it produced in F.

ABDC, FEDC are equal, standing on same base and between same parallels;

but
$$\text{area of FEDC} = CD \times DE$$
$$= a \cdot p;$$
$$\therefore \text{area of ABDC} = a \cdot p.$$

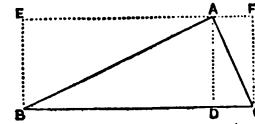
Cor.—Let $AC = b$, and perpendicular on it from $B = q$, we have

$$\text{area} = b \cdot q;$$
$$\therefore a \cdot p = b \cdot q;$$
$$\therefore a : b :: q : p.$$

THE TRIANGLE.

PROP. III.—*To find the area of a triangle, having given the base and the perpendicular height.*

Let BC = a, and AD = p. Through A draw EF parallel to BC, and through B and C draw BE and CF parallel to AD.



Area of rectangle BCFE = $a \cdot p$;
$\therefore$ area of triangle ABC = $\frac{1}{2}a \cdot p$.

Interpreting this, we find the area of the triangle by *multiplying half the base by the perpendicular height.*

Cor. 1.—Let q denote the perpendicular from B on AC. Then

area of triangle = $\frac{1}{2}$AC $\cdot q$;
$\therefore$ $\frac{1}{2}$BC $\cdot p = \frac{1}{2}$AC $\cdot q$;
$\therefore$ BC : AC :: q : p.

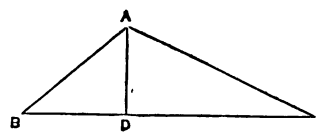
Cor. 2.—A triangle is equivalent to one-half of any parallelogram, having the same base and the same altitude.

Cor. 3.—Triangles having equal bases and equal altitudes are equal.

Cor. 4.—Triangles having equal altitudes are to each other as their bases; triangles having equal bases are to each other as their altitudes, and two triangles are to each other as the products of their bases by their altitudes.

PROP. IV.—*To find the area of a triangle, when the three sides are given.*

In the triangle ABC, let BC = a, AC = b, AB = c. It is required to find the area.



Let p denote the perpendicular from A on BC. Then, by the last proposition,

area of triangle = $\frac{1}{2}ap$.

If, then, we know p in terms of the data, we can find the area of the triangle.

But $p = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$ ... (I. II.)

$$\therefore \text{ area of triangle} = \frac{a}{2} \times \frac{2}{a} \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}; \quad \ldots \quad (1)$$

where s denotes half the sum of the sides of the triangle. Interpreting this formula, we find the area *by taking half the sum of the sides; from this subtracting each side separately, multiplying half the sum and the three remainders together, and taking the square root of the product.*

Remark.—It may here be noticed that each of the factors $(s-a)$, $(s-b)$, $(s-c)$ is a positive quantity, for

$$b + c > a \quad (\textit{Euclid}, \text{I. } 20);$$

$$\therefore a + b + c > 2a;$$

$$\therefore \frac{a+b+c}{2} > a;$$

or $$s > a.$$

Similarly, $$s > b,$$

and $$s > c.$$

Cor. 1.—If the triangle is isosceles, having each of the equal sides equal to a, and the remaining side $= b$, we have, by substituting, in (1), a for c,

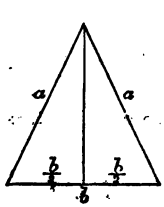
$$\text{area} = \sqrt{s(s-a)(s-a)(s-b)}$$

$$= (s-a)\sqrt{s(s-b)}.$$

But $$s - a = \frac{b}{2} \text{ and } \sqrt{s(s-b)} = \frac{1}{2}\sqrt{4a^2 - b^2};$$

$$\therefore \text{ area} = \frac{b}{4}\sqrt{4a^2 - b^2}. \quad \ldots \quad \ldots \quad (2).$$

A similar result would have been obtained for the area, by first finding the perpendicular p, and recollecting that the perpendicular in this case bisects the base.



$$p = \sqrt{a^2 - \frac{b^2}{4}} = \frac{1}{2}\sqrt{4a^2 - b^2}.$$

$$\therefore \text{ area} = \frac{b}{4}\sqrt{4a^2 - b^2} \quad \ldots \quad \ldots \quad (2).$$

Cor. 2.—If the triangle is equilateral, each side being equal to a, we have, by substituting a for b in (2),

$$\text{Area} = \frac{a}{4}\sqrt{4a^2 - a^2} = \frac{a}{4}\sqrt{3a^2}$$

$$= \frac{a^2}{4}\sqrt{3} \quad \ldots \quad \ldots \quad (3).$$

Interpreting this formula, we find the area of an equilateral triangle *by taking one-fourth of the square of the side, and multiplying it by the square root of* 3.

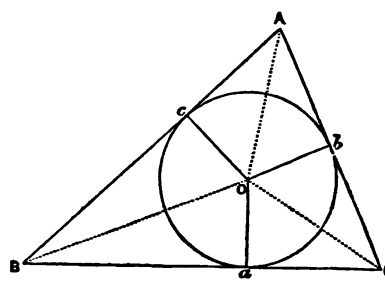
Conversely, if S denotes the area of an equilateral triangle, we can find the length of its side,

for $$S = \frac{a^2}{4}\sqrt{3};$$

$$\therefore \frac{4S}{\sqrt{3}} = a^2;$$

$$\therefore a = \frac{2\sqrt{S}}{\sqrt[4]{3}}.$$

Cor. 3.—Hence we can find r the radius of the



inscribed circle. Since the triangle

$$ABC = BOC + AOC + AOB$$

$$= \frac{a}{2}r + \frac{b}{2}r + \frac{c}{2}r;$$

$$\therefore ABC = \frac{a+b+c}{2} \times r = s \cdot r;$$

$$\therefore r = \frac{\text{area of ABC}}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}.$$

Let S denote the area of the triangle;

$$\therefore r = \frac{S}{s} \quad \ldots \quad \ldots \quad (4).$$

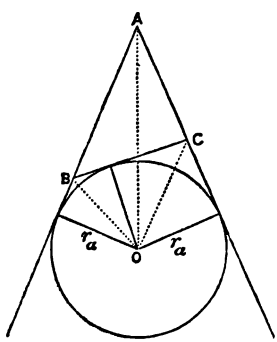
Interpreting this, we find the radius of the inscribed circle *by dividing the area of the triangle by half the sum of the sides.*

Cor. 4.—We can also find the radius of any of the escribed circles; for example, that of the circle touching AB and AC internally and BC externally. Let r_a denote the radius of this circle, we have

triangle $\quad ABC = AOC + AOB - BOC;$

or $$S = \frac{b}{2}r_a + \frac{c}{2}r_a - \frac{a}{2}r_a;$$

$$\therefore r_a = \frac{2S}{b+c-a} = \frac{2S}{2(s-a)} = \frac{S}{(s-a)}.$$



Similarly, taking r_b to denote the radius of the circle touching side b externally, and the other two sides internally, we have

$$r_b = \frac{S}{(s-b)},$$

and $$r_c = \frac{S}{(s-c)}.$$

Cor. 5.—In (I. v.), it was found that the diameter of circumscribing circle is

$$\frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}};$$

$\therefore$ denoting the radius by R, we have

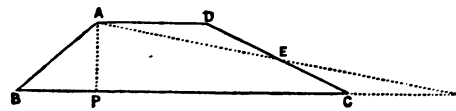
$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} = \frac{abc}{4 \text{ area}}.$$

Interpreting this formula, we find the radius of the

circumscribing circle, *by dividing the product of the sides by four times the area.*

PROP. V.—*To find the area of a trapezoid, having given the altitude and the parallel sides.*

Let ABCD be the trapezoid. Bisect DC in E, join



AE, and produce it to meet BC produced in F. The triangle AED is evidently equal to the triangle ECF, and AD = CF;

$\therefore$ the triangle ABF = trapezoid ABCD.

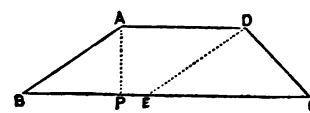
But area of triangle ABF $= \frac{1}{2}$ BF · AP

$= \frac{1}{2}$ (BC + AD)AP ;

$\therefore$ area of trapezoid $= \frac{1}{2}$ (BC + AD)AP.

Interpreting this formula, we find the area of a trapezoid by *multiplying half the sum of the parallel sides by the altitude.*

The same result might have been obtained by dividing the trapezoid into a parallelogram, DB, and a triangle, DEC, by drawing DE parallel to AB.



Area of trapezoid ABCD = area of parallelogram + area of triangle AEC

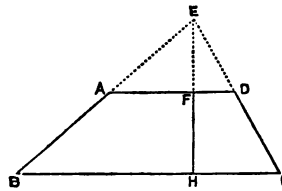
$= BE \cdot AP + \frac{1}{2}EC \cdot AP$

$= AD \cdot AP + \frac{1}{2}(BC - AD)AP$

$= \frac{1}{2}AD \cdot AP + \frac{1}{2}BC \cdot AP$

$= \frac{1}{2}(BC + AD)AP.$

Or we might obtain the area of the trapezoid ABCD by considering it as the difference between the two triangles EBC and EAD.



Let the height FH of the trapezoid $= h$.

Let $EF = x$;

$\therefore x : x + h :: AD : BC$;

$\therefore x \cdot BC = AD\,(x + h)$;

$\therefore x(BC - AD) = h \cdot AD$;

$$\therefore x = \frac{AD}{BC - AD} \times h = EF;$$

$$\therefore EH = \frac{BC}{BC - AD} \times h.$$

Now, area of triangle $EBC = \frac{BC}{2} \times \frac{BC}{BC - AD} \times h$;

and area of triangle $EAD = \frac{AD}{2} \times \frac{AD}{BC - AD} \times h$;

$$\therefore \text{area of trapezoid} = \frac{h}{2(BC - AD)}\{BC^2 - AD^2\}$$

$$= \frac{h}{2}(BC + AD), \text{ as before.}$$

Cor. 1.—If $AD = BC$,

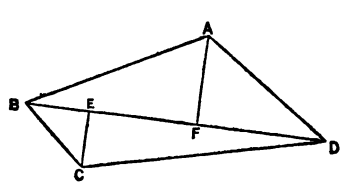
area of $ABCD = \frac{1}{2}(BC + BC)AP = BC \cdot AP$,

as it should be, for the figure is then a parallelogram.

Cor. 2.—If $AD = o$, area of $ABCD = \frac{1}{2}BC \cdot AP$,

as it should be, for the figure is then a triangle.

PROP. VI.—*To find the area of a four-sided figure, having given one diagonal, and the perpendiculars on it from the opposite angles.*



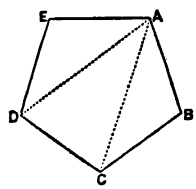
In the figure ABCD, let BD, the diagonal, be equal to D, and the perpendiculars on it from the opposite angles AF and CE be denoted by p and q.

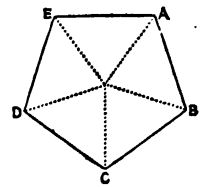
$$\text{Area of triangle ADB} = \tfrac{1}{2}\text{BD} \cdot p,$$
$$\text{area of triangle BCD} = \tfrac{1}{2}\text{BD} \cdot q;$$
$$\therefore \text{area of whole figure} = \tfrac{1}{2}\text{BD} \cdot p + \tfrac{1}{2}\text{BD} \cdot q$$
$$= \tfrac{1}{2}\text{BD}\{p + q\}$$
$$= \frac{\text{D}}{2}\{p + q\}.$$

Hence the area of the figure is found by *multiplying half the diagonal by the sum of the perpendiculars.*

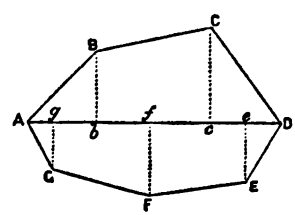
PROP. VII.—*To find the area of any polygon.*

Let it be required to find the area of the polygon ABCDE. We decompose the figure into triangles. We then find the area of each triangle separately, and





take the sum of these areas; this sum is the area of the polygon. We may decompose the polygon either by drawing diagonals from any of the angles, or by joining each of the angles with a point taken within the figure. In the one case, we have as many triangles as there are sides in the figure, less two; in the other case, we have as many triangles as there are sides.



It is sometimes advantageous to adopt the following method: Drawing the greatest diagonal AD, on this drop the perpendiculars from the points B, C, E, F, and G; we thus divide the figure into triangles and trapezoids, the

areas of which we can find, and the sum of these areas is the area of the polygon.

Exercises (4).

1. Find the area of a rectangle whose sides are

(1.) 9 and 8.
(2.) 96 and 48.
(3.) 365 and 400.
(4.) 750 ft. and 621 ft.
(5.) 876 ft. and 940 ft.
(6.) 800 ft. and 910 ft.
(7.) 12 ft. 5 in. and 8 ft. 2 in.
(8.) 14 ft. 3 in. and 11 ft. 9 in.
(9.) 25 yds. 2 ft. and 4 ft. 6 in.

2. Find the area of a square in which the side is

(1.) 110 inches.
(2.) 345 feet.
(3.) 561 yards.
(4.) 871 links.
(5.) 125 links.
(6.) 1·45 chain.

3. Find the area of a rectangle in which the diagonal and side are

(1.) 120 and 75.
(2.) 351 and 108.
(3.) 460 and 230.
(4.) 1760 and 1008.
(5.) 72 and 48.
(6.) 500 and 100.

4. Find the area of a square in which the diagonal is

(1.) 110 yards.
(2.) 275 yards.
(3.) 73·6 yards.
(4.) 1112 yards.
(5.) 1760 yards.
(6.) $30\frac{1}{4}$ yards.

5. The area of a rectangle is 540 square feet, and one of its sides is 12 feet; find the other side.

6. A square contains 112 square yards; find the length of its side.

7. The area of a square is 1260; find its diagonal.

8. Find the area in square yards, feet, and inches of rectangles whose sides are

(1.) 15 feet 9 inches by 12 feet 4 inches.
(2.) 108 feet 5 inches by 17 feet 9 inches.
(3.) 29 feet 8 inches by 20 feet 4 inches.
(4.) 20 feet 6 inches by 20 feet 6 inches.
(5.) 10 yards 2 feet 5 in. by 6 yards 1 foot 8 in.
(6.) 15 yards 1 foot 10 inches by 9 yards 9 inches.

9. The diagonal of a square is 9 feet 9 inches; find its area in square yards, feet, and inches.

10. Find the number of square feet in a board whose length is 15 feet 6 inches, and breadth 10 inches.

11. Find the number of square yards in a floor which is 20 feet 8 inches by 15 feet 6 inches.

12. Find the number of square yards in a square whose side measures 1 pole.

13. A black-board measures 5 feet 4 inches by 4 feet 6 inches; find its area in square yards, feet, and inches.

14. Find the number of acres in a square mile.

15. Find what length of carpet $\frac{3}{4}$ yard wide will cover a floor which measures

(1.) 20 feet by 15 feet.

(2.) 25 feet by 20 feet.

(3.) 36 feet 9 inches by 24 feet 6 inches.

(4.) 24 feet 10 inches by 16 feet 6 inches.

16. Find the cost in each of the above cases when the carpet is 5*s*. 6d. per yard.

17. Find the breadth of a carpet of which 18 yards are required to cover a floor 15 feet by 12 feet.

18. Find what length must be cut from a 10-inch plank, that the area of the portion cut off may be 5 square feet.

19. A postage-stamp measures $\frac{7}{8}$ inch by $\frac{3}{4}$ inch; how many will it take to cover a wall 16 feet by 14 feet 6 inches?

20. How many slates will it take to cover a roof 22 feet 6 inches by 14 feet 8 inches? Each slate measures 1 foot by 9 inches.

21. Find the cost of paving a street $\frac{3}{4}$ths of a mile long by 25 yards broad, at $4\frac{1}{2}$d. per square yard.

22. The sum of the diagonal and side of a square is S; find its area.

23. The difference between the diagonal and side of a square is D; find the area.

24. The diagonal of a rectangle is D, and the difference between two adjacent sides is d; find the area of the rectangle.

25. Find the area of a parallelogram, having given

(1.) Base equal to 100 yards; height, 10 yards.
(2.) Base equal to 14 feet 3 inches; height, 2 feet 9 inches.

26. A field in the form of a parallelogram, whose base measures 120 yards, contains $1\frac{1}{2}$ acre; find the height.

27. Two adjacent sides of a parallelogram are 1200 and 860 yards; the perpendicular on the former from the opposite side is 120; find the perpendicular on the latter from the opposite side.

28. Find the area of a triangle of which the sides are

(1.) 7, 8, 9.
(2.) 12, 15, 18.
(3.) 64, 64, 32.
(4.) 475, 574, 745.
(5.) 1000, 1100, 1200.
(6.) 1760, 2350, 3000.
(7.) 4·5, 6·2, 7·8.
(8.) 10·2, 12·5, 12·5.
(9.) 15·45, 20·1, 25·5.
(10.) 12, 12, 12.
(11.) ·012, ·012, ·012.
(12.) ·9, 1·0, 1·1.

29. Find the area in acres, roods, and poles of a triangle in which the sides are

(1.) 456 links, 512 links, 600 links.
(2.) 1200 links, 1450 links, 1500 links.
(3.) 12 chains 45 links, 15 chains 70 links, 16 chains 25 links.

30. The perimeter of a triangle is 120 feet, and the sides are proportional to 5, 9, and 10; find the area.

31. Find the radii of the circles inscribed in the triangles whose sides are

(1.) 45, 50, 60.
(2.) 110, 120, 130.
(3.) 450, 620, 840.

32. In the above triangles, shew how to find the radii of the circles which touch the least sides externally, and the other two internally.

33. Compare the radii of the inscribed and circumscribed circles when the sides of the triangle are

(1.) 4, 5, 6.
(2.) 3, 3, 3.

34. If S denote the area of a triangle r, r_a, r_b, r_c, the

radii of the inscribed and escribed circles, prove $S = \sqrt{rr_a r_b r_c}$.

35. Shew that in a right-angled triangle the diameter of the circumscribing circle is equal to the hypothenuse.

36. Find the area of a trapezoid, of which the parallel sides are 25 and 15 inches, and the perpendicular distance between them 1 foot.

37. Find the area of a four-sided figure ABCD, of which AB = 28 yards, BC = 45 yards, CD = 51 yards, DA = 52 yards, and the diagonal AC = 53 yards.

38. The diagonals of a rhombus are 88 yards and 110 yards; find the area.

39. Shew how to find the area of a regular hexagon inscribed in a circle whose radius is 1 foot.

40. Shew how to find the area of a regular dodecagon inscribed in a circle whose radius is 1 foot.

41. Shew how to find the area of a regular pentagon inscribed in a circle whose radius is 1 foot.

42. A square and an equilateral triangle have the same area; compare their perimeters.

43. A square and an equilateral triangle have the same perimeters; compare their areas.

SIMILAR TRIANGLES.

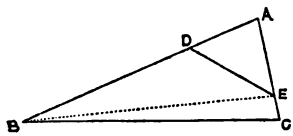
Def.—Two polygons are similar when they are equiangular, and have also the sides about the equal angles proportional. In the case of triangles, either of the above conditions is sufficient, as the one condition implies the other (*Euclid*, VI. 4, 5), and hence we say that:

Similar triangles are those which are equiangular, or those which have their sides proportional.

It will follow, from the above definition, that triangles are similar when they have their sides *parallel* each to each, or *perpendicular* each to each.

PROP. VIII.—*Two triangles which have an angle of the one equal to an angle of the other, are to each other as the products of the sides including the equal angles.*

The triangles which have an angle of the one equal to an angle of the other, may be placed with their equal angles in coincidence.



Let ABC and ADE be the two triangles having the common angle A.

Then $$\frac{\text{area of ABC}}{\text{area of ADE}} = \frac{\text{AB} \times \text{AC}}{\text{AD} \times \text{AE}}.$$

Join BE.

Then we have $$\frac{\text{ABC}}{\text{ABE}} = \frac{\text{AC}}{\text{AE}} \quad \ldots \quad (1),$$

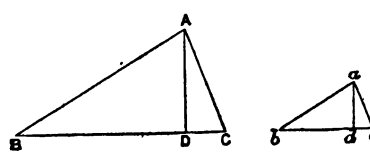
and $$\frac{\text{ABE}}{\text{ADE}} = \frac{\text{AB}}{\text{AD}} \quad \ldots \quad (2).$$

Multiplying (1) and (2) together, we have

$$\frac{\text{ABC}}{\text{ADE}} = \frac{\text{AB}}{\text{AD}} \times \frac{\text{AC}}{\text{AE}} \quad \ldots \quad (3).$$

Hence the truth of the proposition.

PROP. IX.—*The areas of similar triangles are to each other as the squares of their like sides.*



Let ABC, *abc*, be two similar triangles, to prove

$$\frac{\text{area of ABC}}{\text{area of } abc} = \frac{\text{BC}^2}{bc^2} = \frac{\text{AB}^2}{ab^2} = \frac{\text{AC}^2}{ac^2},$$

$$\text{area of ABC} = \tfrac{1}{2}\text{BC} \cdot \text{AD},$$

$$\text{area of } abc = \tfrac{1}{2}bc \cdot ad,$$

$$\therefore \frac{\text{area of ABC}}{\text{area of } abc} = \frac{\text{BC}}{bc} \times \frac{\text{AD}}{ad} \quad \ldots \quad (1).$$

But the two triangles ABD, *abd*, are also similar;

$$\therefore \frac{\text{AD}}{ad} = \frac{\text{AB}}{ab}. \quad \text{But } \frac{\text{AB}}{ab} = \frac{\text{BC}}{bc};$$

$$\therefore \frac{AD}{ad} = \frac{BC}{bc}.$$

Substituting this value of $\frac{AD}{ad}$ in (1), we have

$$\frac{\text{area of ABC}}{\text{area of } abc} = \frac{BC}{bc} \cdot \frac{BC}{bc} = \frac{BC^2}{bc^2} = \frac{AB^2}{ab^2} = \frac{AC^2}{ac^2},$$

which proves the proposition.

Cor.—We might have deduced this proposition directly from the preceding, for

$$\frac{\text{area of ABC}}{\text{area of } abc} = \frac{AB \times AC}{ab \times ac}.$$

But since the triangles are similar,

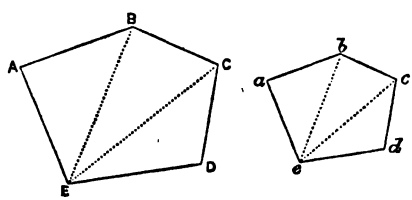
$$\frac{AC}{ac} = \frac{AB}{ab};$$

$$\therefore \frac{\text{area of ABC}}{\text{area of } abc} = \frac{AB^2}{ab^2}.$$

PROP. X.—*The areas of similar polygons are to each other as the squares of their like sides.*

Let ABCDE, *abcde*, be two similar polygons, denoting their areas by S and *s*, it is required to prove

$$\frac{S}{s} = \frac{AB^2}{ab^2} = \frac{BC^2}{bc^2} = \&c.$$



For let the polygons be decomposed into similar triangles, then

$$\frac{ABE}{abe} = \frac{EBC}{ebc} = \frac{ECD}{ecd} = \frac{AB^2}{ab^2};$$

$$\therefore \frac{ABE + EBC + ECD}{abe + ebc + ecd} = \frac{AB^2}{ab^2},$$

$$\text{or } \frac{S}{s} = \frac{AB^2}{ab^2},$$

which proves the proposition.

Cor.—Since circles are similar figures, it follows that the *areas of two circles are to each other as the squares of their circumferences, or of their diameters.*

Exercises (5).

1. The sides of a triangle are 9, 10, 11; find the area of the triangle cut off by a line parallel to the longest side, and equal to one-third of it.

2. The sides of a triangle are 15, 18, 21; find the sides of a similar triangle whose area is 100.

3. The sides of a triangle are 25, 51, 74; find the sides of a similar triangle whose area is $2133\frac{1}{3}$.

4. The sides of a rectangle whose area is 2 acres 3 roods 22 poles are in the proportion of 2 to 3; find the sides.

5. The base and perpendicular of a right-angled triangle are 28 and 45; find the sides of a similar triangle which contains only half the area.

6. The sides of a triangle are 7, 12, 17; find the area of a similar triangle whose *greatest* side is 59; find also the area of a similar triangle whose *least* side is 59.

7. The sides of a triangle are proportional to the numbers 13, 14, 15, and the area is 24336 square feet; find the length of the sides.

8. The circumference of a circle is 10 yards; find the diameter of another circle which is 3 times larger.

9. Compare the areas of two circles whose radii are proportional to 3 and 5.

10. The areas of two circles are 120 and 84; compare their radii.

11. A field containing *one* half-acre is laid down on a plan to a scale of 1 inch to 20 feet; find how much paper the plan will cover.

12. The plan of a field containing 5 acres 3 roods

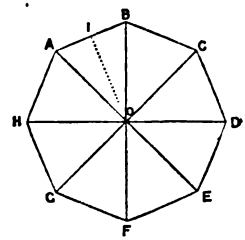
20 poles covers 25 square inches of paper; what is the scale to which the plan has been drawn?

13. Determine the scale used in the construction of a plan, when 1 square inch represents 20 acres.

14. Find the size of a property in acres, whose plan covers a square feet, the scale being 1 inch to 5 yards.

THE AREA OF A CIRCLE.

PROP. XI.—*The area of a regular polygon is equal to the perimeter of the polygon multiplied by half the radius of the inscribed circle.*



Taking for example the regular figure ABCDEFGH. Join the centre, O, of the polygon with all the angles; we thus decompose the figure into eight triangles equal to each other.

$$\text{Area of triangle AOB} = \frac{AB}{2} \times OI$$

$$\text{Area of triangle BOC} = \frac{BC}{2} \times OI$$

$$\text{Area of triangle COD} = \frac{CD}{2} \times OI$$

$$\text{Area of triangle DOE} = \frac{DE}{2} \times OI$$

$$\&c. \qquad \&c.$$

$$\vdots \qquad \vdots$$

$$\text{Area of triangle HOA} = \frac{HA}{2} \times OI$$

$\therefore$ Area of whole polygon

$$= \tfrac{1}{2}(AB + BC + CD + \ldots + HA) \times OI$$

$$= \frac{\text{perimeter}}{2} \times \text{radius of inscribed circle,}$$

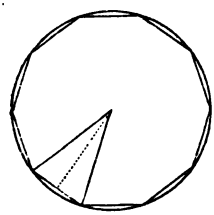
or

$$= \text{perimeter} \times \frac{\text{radius of inscribed circle}}{2}.$$

PROP. XII.—*The area of a circle is equal to the circumference multiplied by half the radius.*

In a circle of radius R, inscribe a regular polygon of n sides.

The area of polygon $= \frac{\text{perimeter}}{2} \times$ radius of inscribed circle.



If the number of sides of the polygon be indefinitely increased, it follows that

(1.) The area of the polygon will approach nearer and nearer the area of the circle, and will ultimately be equal to it.

(2.) The perimeter of the polygon will ultimately become the circumference of the circle.

(3.) The radius of the inscribed circle will ultimately become the radius of the given circle.

Therefore the above formula will become

$$\text{Area of circle} = \frac{\text{circumference}}{2} \times \text{radius}$$

$$= \text{circumference} \times \frac{\text{radius}}{2}.$$

That is, the area of the circle *is the product of half the circumference by the radius.*

Cor. 1.—Since area of circle $= \text{circumference} \times \frac{\text{radius}}{2}$

$= \frac{1}{2}$ circumference $\times$ radius,

denoting the radius by r, diameter by d, circumference by c, and area by A, we have

$$A = \tfrac{1}{2} c \cdot r \quad \ldots \quad \ldots \quad (1),$$

or $$= \pi r^2 \quad \ldots \quad \ldots \quad (2).$$

Interpreting this last result, we find the area of the circle *by multiplying the square of the radius by the constant multiplier* π.

It may be noted that $\pi = 3{\cdot}14159$ or $3\frac{1}{7}$.

Cor. 2.—Area of circle $= \pi r^2$,

or $$A = \pi\left(\frac{\text{diameter}}{2}\right)^2 = \frac{\pi}{4}d^2.$$

Cor. 3.—Since area $= \pi r^2$;

$$\therefore r^2 = \frac{\text{area}}{\pi} = \frac{A}{\pi};$$

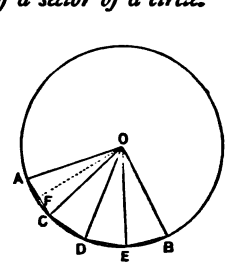
$$\therefore r = \sqrt{\frac{\text{area}}{\pi}} = \frac{1}{\sqrt{\pi}}\sqrt{\text{area}} = \frac{\sqrt{A}}{\sqrt{\pi}},$$

$$d = \frac{2}{\sqrt{\pi}}\sqrt{\text{area}} = \frac{2}{\sqrt{\pi}}\sqrt{A},$$

$$c = 2\sqrt{\pi\ \text{area}} = 2\sqrt{\pi A}.$$

PROP. XIII.—*To find the area of a sector of a circle.*

Def.—A sector of a circle is a portion of the circle bounded by two radii and an arc.



Let AOB be the sector. Divide the arc AB into a certain number of equal parts. Drawing the chords of these arcs, we thus form the equal triangles AOC, COD, &c.

Area of AOC $= \frac{1}{2}$ AC · OF.

Conceive the number of triangles to augment continually, the sum of their areas approaches nearer and nearer to the area of the sector, and ultimately coincides with it.

Hence area of sector $= \frac{1}{2}$ arc × radius.

Interpreting this, we find the area of the sector by *multiplying half its arc by the radius of the circle.*

and $$aA = R - r;$$

$$\therefore Ba \cdot aA = (R + r)(R - r).$$

But $$Ba \cdot aA = Ca^2;$$

$$\therefore (R + r)(R - r) = Ca^2;$$

$$\therefore \pi(R + r)(R - r) = \pi Ca^2;$$

$\therefore$ *area of ring is equal to that of a circle whose diameter is a chord of the exterior circle touching the interior.*

Exercises (6).

1. Find the area of the circle whose radius is

(1.) 40 feet.
(2.) 20·5 feet.
(3.) 150 yards.
(4.) 1760 yards.
(5.) 1 inch.
(6.) 5 feet 9 inches.

2. Find the area of the circle whose circumference is

(1.) 500 feet.
(2.) 200 yards.
(3.) one half-mile.
(4.) 10 feet.
(5.) 3·14159 feet.
(6.) 60 yards.

3. Find the radius of the circle whose area is

(1.) 50 square feet.
(2.) 1 square mile.
(3.) 240 square feet.
(4.) one half-acre.
(5.) 2 ac. 3 ro. 10 po.
(6.) 1760 square yards.

4. Find the area of a circle whose diameter is 14 links.

5. Find the diameter of a circle whose area is 45 square yards 3 square feet.

6. Find the length of the radius when the area is 1 acre.

7. Find the radius of a circle of which the area is the same as that of an equilateral triangle whose side is 5 feet.

8. The radius of a circle is 10 feet, and a square is inscribed within it; find the difference between the area of the circle and that of the square.

9. A circle is inscribed within a triangle whose sides are 25, 30, 35; find the difference between the area of the triangle and that of the circle.

10. The area of a square being equal to that of a circle; compare their perimeters.

11. The sides of a triangle are 30, 40, 50; find the area of a circle which has the same perimeter.

12. The radius of a circle is 10 feet; find the radius of a circle which is ten times as large.

13. Find the total pressure on a circular plate of 12 feet in radius, the pressure on each square inch being 15 lbs.

14. Find the area of a sector of a circle whose radius is 21 feet, the length of the arc being 15 feet.

15. In a circle whose radius is 12 feet, what must be the length of the arc, so that the area of the sector will be 100 square feet?

16. Find the area of the portion between the two concentric circles whose radii are 12 and 10 feet.

17. A circle of 10-inch radius is to be divided into three equal parts by concentric circles; find the length of their radii.

18. The radius of the inner boundary of a ring is 14 inches, the area of the ring is 100 square inches; find the radius of the outer boundary.

19. A circular grass plot contains 1760 square yards, it is surrounded by a gravel walk 10 feet broad; find the area of the walk.

20. What is the radius of that circle in which a sector of 75° contains 120 square feet?

21. The diameter of a circle is 30 inches; find the area of a segment cut off by a chord of 24 inches.

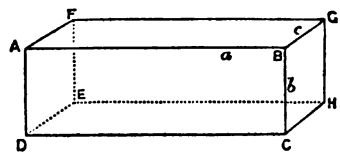
22. The radius of a circle is 6 feet; find the area of the sector, and the length of the arc which contains 115°.

23. Find the radius of a circle whose area is (1) equal to the sum of the areas of two circles of 12 and 15 inches, (2) equal to the difference of the areas of two circles of 12 and 15 inches.

II.—SURFACES OF SOLIDS.

PROP. I.—*To find the surface of a rectangular parallelopiped, whose coterminous edges are a, b, c.*

Def.—A polyedron is a geometrical solid bounded by plane faces.



A parallelopiped is a polyedron each of whose faces is a parallelogram.

A right parallelopiped is a parallelopiped whose lateral edges are perpendicular to the planes of its bases. Hence the lateral faces are rectangles, but its bases may be either rhomboids or rectangles.

A rectangular parallelopiped is a right parallelopiped whose bases are rectangles. Hence it is a parallelopiped each of whose faces is a rectangle.

A *cube* is a rectangular parallelopiped whose six faces are squares.

There are here three pairs of rectangles.

$$\left.\begin{array}{l}\text{Area of face ABCD} = ab\\ \text{and area of face FEHG} = ab\end{array}\right\}$$

$$\left.\begin{array}{l}\text{Area of face ADEF} = bc\\ \text{and area of face BCHG} = bc\end{array}\right\}$$

$$\left.\begin{array}{l}\text{Area of face ABGF} = ac\\ \text{and area of face DEHC} = ac\end{array}\right\};$$

$$\therefore \text{ total surface} = 2(ab + bc + ac).$$

Hence the rule.

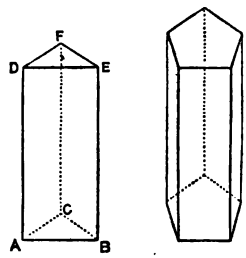
Cor. 1.—If $a = b = c$, the figure is then a cube;

$$\therefore \text{ surface of cube} = 6a^2.$$

Hence, to find the surface of a cube, *square the edge, and take six times the result.*

PROP. II.—*To find the surface of a right prism.*

Def.—A prism is a polyedron two of whose faces are equal polygons, lying in parallel planes, and having their homologous sides parallel, the other faces being parallelograms formed by the intersections of planes passed through the homologous sides of the equal polygons. A right prism is one whose lateral edges are perpendicular to the planes of its bases.



The total surface is composed of the lateral surface and the surface of the two ends.

The lateral surface is made up of the areas of rectangles, and the surface of the ends by that of triangles or polygons which we can calculate.

THE RIGHT CIRCULAR CYLINDER.

PROP. III.—*The lateral surface of a right circular cylinder is the product of the circumference of the base by the height.*

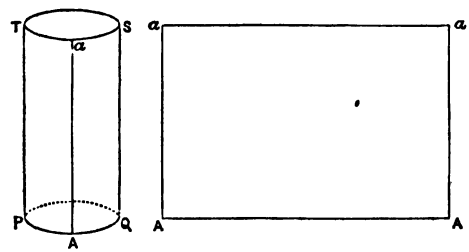
Def.—A cylindrical surface is a curved surface generated by a moving straight line, which continually touches a given curve, and in all its positions is parallel to a given fixed straight line not in the plane of the curve.

The moving line is called the *generatrix*, the curve which it touches is called the *directrix*. Any straight line in the surface which represents one of the positions of the generatrix is called an *element* of the surface.

The solid bounded by a cylindrical surface and two parallel planes is called a *cylinder*.

A cylinder whose base is a circle, is called a *circular* cylinder. A *right circular cylinder* is one whose elements are perpendicular to the base.

Let PQST be the cylinder whose surface it is required to find. Conceive the cylinder to be hollow, and let it be slit open along Aa, any position of the generating line. The surface, when spread out, will then take



the form of the rectangle AAaa, whose side AA corresponds to the circumference of the base, and side Aa to the height of the cylinder.

$$\text{Area of AA}aa = \text{AA} \cdot \text{A}a;$$

$\therefore$ lateral surface of cylinder = circumference of base $\times$ height.

Let r denote the radius of the base, and h the height;

$$\therefore \text{ lateral surface} = 2\pi rh.$$

To find the total surface, we must add to this the surface of the two ends;

$$\therefore \text{ total surface} = 2\pi rh + 2\pi r^2,$$

or
$$= 2\pi r\{h + r\} \quad \ldots \quad \ldots \quad (1).$$

Def.—Two cylinders are said to be *similar* when their heights are proportional to their radii.

Cor. 1.—Let S and S′ denote the *lateral* surfaces of two similar cylinders; T and T′, their *total* surfaces.

Then
$$\frac{S}{S'} = \frac{r^2}{r'^2},$$

and
$$\frac{T}{T'} = \frac{r^2}{r'^2}.$$

Since the cylinders are similar,

$$\frac{r}{r'} = \frac{h}{h'} = \frac{r+h}{r'+h'} \quad \ldots \quad \ldots \quad (1).$$

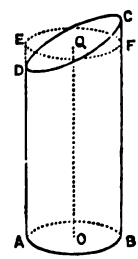
But $$\frac{S}{S'} = \frac{2\pi rh}{2\pi r'h'} = \frac{rh}{r'h'} = \frac{r}{r'} \cdot \frac{h}{h'} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}.$$

Again, $$\frac{T}{T'} = \frac{2\pi r(r+h)}{2\pi r'(r'+h')} = \frac{r(r+h)}{r'(r'+h')} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}.$$

It follows, therefore, that *the lateral surfaces, or the total surfaces of similar cylinders, are to each other as the squares of their radii, or as the squares of their heights.*

Cor. 2.—If $h = r$, that is, if the height be equal to the radius of the base, the lateral surface is equal to the area of the two ends. If the $h = 2r$, the lateral surface is equal to twice the area of the ends.

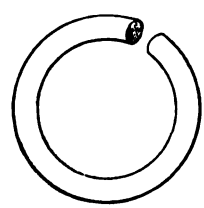
Hence $$\frac{\text{lateral surface}}{\text{area of the two ends}} = \frac{\text{height of cylinder}}{\text{radius of base}}.$$



Cor. 3.—If the cylinder be cut by a plane which is not parallel to the base, the lateral surface will evidently be equal to that of a cylinder, ABFE, on the same base, and whose height, OQ, is half the sum of BC and AD.

PROP. IV.—*To find the surface of a solid ring.*

1. If the ring be of circular section.



The solid ring may be considered as a cylinder bent round till the ends meet; the surface of the ring will therefore be found by multiplying the circumference of the base by the height; the base being a circular section, and the height the mean between the outer and inner circumferences.

Let D and d be the outer and inner diameters of the circular ring.

Then thickness of ring $= \frac{D-d}{2}$;

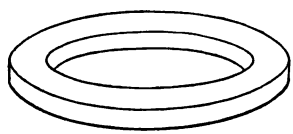
$\therefore$ circumference of circular section $= \pi\left(\frac{D-d}{2}\right)$,

and the mean length or height $= \pi\left(\frac{D+d}{2}\right)$;

$$\therefore \text{surface of ring} = \frac{\pi^2}{4}\{D+d\}\{D-d\}$$

$$= \frac{\pi^2}{4}\{D^2-d^2\} = \tfrac{1}{4}\{(\pi D)^2-(\pi d)^2\} = \tfrac{1}{4}\{C^2-c^2\}.$$

Hence it follows that *the surface of the ring is equal to one-fourth of the difference between the squares on the outer and inner circumferences.*



2. If the ring be flat.

The surface of the ring is made up of:

The upper and lower surfaces, which are equal; the outer or convex surface; and the inner or concave surface.

Let D and d denote the outer and inner diameters, and h the height.

Upper surface $= \frac{\pi}{4}(D^2-d^2)$;

$\therefore$ upper and lower surfaces $= \frac{\pi}{2}(D^2-d^2)$.

Outer or convex surface $= \pi Dh$.

Inner or concave surface $= \pi dh$;

$$\therefore \text{total surface} = \frac{\pi}{2}\{D^2-d^2\} + \pi Dh + \pi dh$$

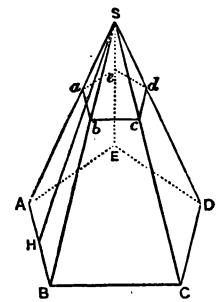
$$= \frac{\pi}{2}\{D+d\}\{D-d+2h\}.$$

Hence, to find the surface, we have the following

Rule.—Add twice the height to the difference of the

diameters, and multiply the sum by the sum of the diameters and by $\frac{\pi}{2}$.

PROP. V.—*The lateral surface of a regular pyramid is equal to the product of the perimeter of the base, by one-half of the slant height.*



Def.—A pyramid is a polyedron bounded by a polygon and triangular faces, formed by the intersections of planes passed through the sides of the polygon and a common point out of its plane, as S – ABCDE.

When the sides of the bounding polygon are equal, the pyramid is *regular.*

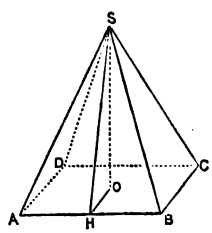
A frustum of a pyramid is the portion of the pyramid included between the base and a plane parallel to the base, and cutting all its lateral edges.

Let S – ABCDE be a regular pyramid, the lateral faces, SAB, SBC, &c. being equal isosceles triangles, whose bases are the sides of the regular polygon ABCDE, and whose common altitude is the slant height SH, the sum of their areas, or the lateral area of the pyramid, is equal to

$$AB + BC + CD + \&c. \times \tfrac{1}{2}SH,$$

or perimeter $\times \frac{1}{2}$ slant height.

Example.—1. Let it be required to find the lateral surface of a square pyramid, the side of whose base is s, and slant height h.



Lateral surface $= 2sh$.

2. If the side $= s$, and the edge SB $= k$.

Then $$SH = \sqrt{k^2 - \frac{s^2}{4}}$$

$$= \tfrac{1}{2}\sqrt{4k^2 - s^2};$$

$$\therefore \text{ lateral surface} = 2s \cdot \tfrac{1}{2}\sqrt{4k^2 - s^2}$$
$$= s\sqrt{4k^2 - s^2}.$$

3. If the side $= s$, and perpendicular height SO $= p$.

Then $$\text{SH} = \sqrt{p^2 + \frac{s^2}{4}} = \tfrac{1}{2}\sqrt{4p^2 + s^2};$$

$$\therefore \text{ lateral surface} = s\sqrt{4p^2 + s^2}.$$

Cor.—The lateral surface of the frustum of a regular pyramid is equal to half the sum of the perimeters of its bases multiplied by slant height of the frustum.

For this product is the measure of the sum of the trapezoids AB*ba*, BC*cb*, &c. whose common altitude is the slant height *h*H.

THE RIGHT CIRCULAR CONE.

PROP. VI.—*The lateral surface of a right circular cone is equal to the circumference of the base multiplied by half the slant height.*

Def.—A conical surface is a curved surface generated by a moving straight line which continually touches a given curve, and passes through a given fixed point not in the plane of the curve. The moving line is called the *generatrix*, and the curve which it touches is the *directrix*.

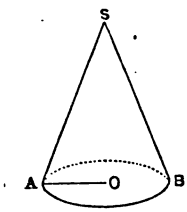
Any straight line in the surface which represents one of the positions of the generatrix is called an *element* of the surface.

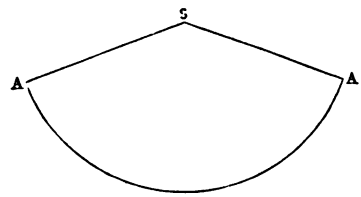
The solid bounded by a conical surface, and a plane cutting the surface, is called a *cone*. The plane surface is called the base.

A cone whose base is a circle is called a *circular* cone. The straight line joining the fixed point or *vertex* of the circular cone to the centre of its base is called the *axis* of the cone.

A *right circular cone* is a circular cone whose axis is perpendicular to its base.

Let SAB be the cone whose surface it is required to find. Conceive the cone to be hollow, and let it be slit open along the line SA, any position of the generatrix; when spread out, it will form a sector of a circle whose area is found (II. XIII.) by multiplying the arc by half the radius. But the arc of the sector is the





circumference of the base of the cone, and the radius of the circle is the slant height of the cone;

$\therefore$ lateral surface of cone

$$= \text{circumference of base} \times \frac{\text{slant height}}{2}.$$

If r = radius of base, and h = slant height, we have

$$\text{lateral surface} = \pi rh.$$

If to this we add the area of the base, that is, πr^2, we will have

$$\text{total surface} = \pi r^2 + \pi rh = \pi r\{r + h\}.$$

Cor. 1.—Hence, as in the case of similar cylinders, we can prove that *the lateral surfaces of similar cones are to each other as the squares of their slant heights, or as the squares of the radii of their bases.*

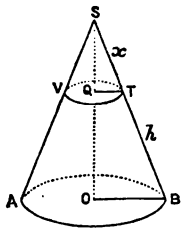
Cor. 2.—If the perpendicular height SO of the cone be given, and the radius of base, we will evidently have

$$\text{lateral surface} = \pi r \sqrt{p^2 + r^2}$$

where p = SO, the perpendicular height.

Def.—The frustum of a cone is that portion of the cone included between its base and a cutting plane parallel to the base.

Cor. 3.—The lateral surface of the frustum of a cone *is equal to half the sum of the circumferences of the ends multiplied by the slant height of the frustum.*



The surface of the frustum TBA is equal to the difference between the surfaces of the two cones SAB and SVT.

Let R and r denote the radii OB and QT, and let BT $= h$.

To find ST, the slant height of the smaller cone, let ST $= x$;

$$\therefore R : r :: h + x : x;$$

$$\therefore x = \frac{rh}{R - r} = ST;$$

$$\therefore SB = \frac{Rh}{R - r}.$$

$$\text{Lateral surface of larger cone} = \pi R \cdot \frac{Rh}{R - r},$$

$$\text{lateral surface of smaller cone} = \pi r \cdot \frac{rh}{R - r};$$

$$\therefore \text{surface of frustum} = \frac{\pi h}{R - r}\{R^2 - r^2\}$$

$$= \pi h\{R + r\} = (\pi R + \pi r)h.$$

But πR denotes half the circumference of one end, and πr denotes half the circumference of other end. Hence the rule.

If to this we add $\pi R^2 + \pi r^2$, the areas of the ends, we have

$$\text{total surface} = \pi\{R^2 + (R + r)h + r^2\}.$$

If in this formula $r = o$, we have, as before, the expression for the total surface of the complete cone.

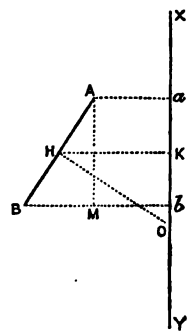
If $r = R$, we have the expression for the total surface of the cylinder, as might have been expected.

PROP. VII. *Lemma.* — *The surface generated by a straight line revolving about an axis in its plane, is equal to the projection of the line on the axis multiplied by the circumference of the circle whose radius is the perpendicular erected at the middle of the line, and terminated by the axis.*

Def.—The *projection* of a point A upon line XY is the foot *a* of the perpendicular let fall from the point upon the line.

The *projection* of *a finite straight line* AB upon the line XY is the distance *ab* between the projections of the extremities of AB.

Let AB be a straight line, revolving about the axis XY, *ab* its projection on the axis, HO the perpendicular to it at its middle point H, terminating in the axis.



To prove that

surface generated by AB = *ab* × circumference, having OH as radius.

Draw AM parallel and HK perpendicular to the axis.

The surface generated by AB is evidently that of a frustum of a cone.

Hence (Prop. VI.),

surface generated by AB = AB × circumference having HK as radius. ... ... (1).

Now, the triangles ABM and OHK are similar, their sides being mutually perpendicular to each other.

Then $AB : AM :: HO : HK,$

or $AB : ab :: HO : HK,$

or $AB : ab :: 2\pi HO : 2\pi HK.$

$\therefore$ AB : *ab* :: circumference having HO as radius : circumference having HK as radius;

$\therefore$ circumference having HK as radius $= \frac{ab}{AB} \times$ circumference having HO as radius.

Substituting in (1) this value for circumference having HK as radius;

$\therefore$ surface generated by AB $= ab \times$ circumference having HO as radius.

Cor.—If AB is parallel to the axis, then the surface generated is that of a cylinder, and *ab* = AB, HO = radius of base.

THE SPHERE.

PROP. VIII.—*The surface of a spherical zone is equal to the product of its altitude by the circumference of the sphere.*

Def.—A *sphere* is a solid bounded by a surface, all points of which are equally distant from a point within it, called the centre.

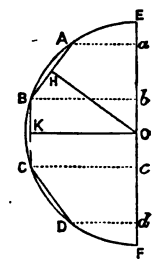
A sphere may be generated by the revolution of a semicircle about its diameter as an axis, for a surface thus generated will evidently have all its points equally distant from the centre.

It can be shewn that every section of a sphere made by a plane is a circle, and as the greatest possible section is one made by a plane passing through the centre, such a section is called a *great circle*, any other a *small circle*.

The *poles* of a circle of the sphere are the extremities of the diameter of the sphere, which is perpendicular to the plane of the circle, and the diameter is called the *axis* of the circle.

A *spherical zone* is a portion of the sphere intercepted between two parallel planes. If one of the planes touch the sphere, the zone is said to be of a single base.

Let the sphere be generated by the revolution of the semicircle EBF about the axis EF, and let the arc AD generate the zone whose area is required.



Let the arc AD be divided into any number of equal parts, AB, BC, CD, &c. The chords AB, BC, CD, &c. form part of the perimeter of a polygon, which differs from a regular polygon only in this, that the arc subtended by one of its sides, as AB, is not necessarily an aliquot part of the whole circumference. The sides being equidistant from the centre, the perpendiculars OH, OK, &c. are equal. Therefore we have, by the preceding Lemma,

Surface generated by AB = ab × circumference of which OH is radius.

Surface generated by BC = bc × circumference of which OH is radius.

Surface generated by CD = cd × circumference of which OH is radius.

Taking the sum, we have

surface generated by AB + BC + CD = $(ab + bc + cd)$ × circumference of which OH is radius.

This being true, whatever be the number of chords AB, BC, &c. let that number be increased indefinitely.

Then the surface generated by the chords approaches indefinitely to the area of the zone generated by the arc AD, and OH approaches nearer and nearer to the radius of the sphere, and ultimately coincides with it; hence at the limit we have

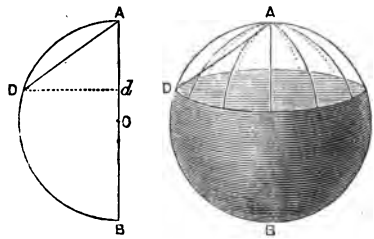
surface of zone = ad × circumference of sphere.

Let S denote the surface of the zone whose altitude is h, the radius of the sphere being r.

We have $S = 2\pi rh$.

Cor. 1.—*Zones of the same sphere are evidently to each other as their altitudes.*

Cor. 2.—Let the arc AD generate a zone of a single



base, its surface $= 2\pi \cdot OA \cdot Ad$

$$= \pi\{BA \cdot Ad\} = \pi \cdot AD^2.$$

That is, *the surface of a zone of one base is equal to that of a circle whose radius is the chord of the generating arc of the zone.*

PROP. IX.—*The surface of a sphere is equal to the product of its diameter by the circumference of the sphere.*

This follows directly from the preceding proposition, since the surface of the whole sphere may be regarded as a zone, whose altitude is the diameter of the sphere.

Cor. 1.—Let S = the surface of sphere whose radius is r.

Then $$S = 2\pi r \cdot 2r = 4\pi r^2;$$

$$\therefore r = \sqrt{\frac{S}{4\pi}} = \frac{1}{2}\sqrt{\frac{S}{\pi}},$$

or *the surface of the sphere is equivalent to that of four great circles.*

Cor. 2.—The total surface of the cylinder circumscribing the sphere whose radius is r, is evidently

$$6\pi r^2;$$

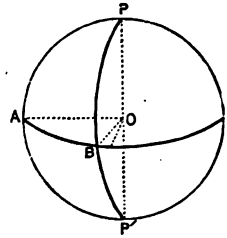
$\therefore$ *surface of sphere* $= \frac{2}{3}$ *of surface of circumscribing cylinder.*

Cor. 3.—Let S and S′ denote the surfaces of two spheres whose radii are r and r'.

Then $$\frac{S}{S'} = \frac{4\pi r^2}{4\pi r'^2} = \frac{r^2}{r'^2}$$

Hence *the surfaces of spheres are to each other as the squares of their radii.*

Def.—A *lune* is that portion of a sphere contained between two semi-circumferences of great circles, and terminated by the same diameter; the solid angle formed by the planes of the semicircles is said to be the angle of the lune.



It is shewn that *two lunes are to each other as their angles;* hence:

PROP. X.—*The ratio of the surface of the lune to the whole surface of the sphere is the same as the ratio of the angle of the lune to four right angles; or, in a sphere whose radius is r, if A denote the ratio of the angle of the lune to a right angle, we have*

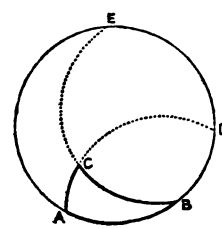
$$\frac{\text{surface of lune}}{\text{surface of sphere}} = \frac{A}{4};$$

$$\therefore \text{surface of lune} = \frac{A}{4} \text{ surface of sphere}$$

$$= A \times \pi r^2.$$

PROP. XI.—*The surface of a spherical triangle is to that of the sphere as the sum of the angles of the triangle, diminished by two right angles, is to eight right angles.*

Let ABC be the spherical triangle. Produce the sides AC and BC, till they intersect the circumference of which the third side AB is a part. We then have



ABC + BCD = lune A,
ABC + ACE = lune B,
ABC + CDE = lune C.

Adding, we have

$$2ABC + ABC + BCD + CDE + ACE = \text{lune A} + \text{lune B} + \text{lune C},$$

or $2ABC + \frac{1}{2}$ surface of sphere = lune A + lune B + lune C;

$$\therefore ABC = \tfrac{1}{2}(\text{lune A} + \text{lune B} + \text{lune C}) - \tfrac{1}{4} \text{ surface of sphere};$$

$$\therefore \frac{ABC}{\text{surface of sphere}} = \frac{\text{lune A}}{2 \text{ surface of sphere}} + \frac{\text{lune B}}{2 \text{ surface of sphere}} + \frac{\text{lune C}}{\text{surface of sphere}} - \tfrac{1}{4}.$$

Denoting the surface of the sphere by S, we have

$$\frac{ABC}{S} = \frac{\text{lune A}}{2S} + \frac{\text{lune B}}{2S} + \frac{\text{lune C}}{2S} - \tfrac{1}{4}.$$

or as in the last proposition,

$$\frac{ABC}{S} = \frac{A + B + C - 2}{8}.$$

Exercises (7).

1. Find the total surface of a parallelopipedon which measures

(1.) 10 inches, 15 inches, 18 inches.
(2.) 2 feet 3 inches, 5 feet 4 inches, 6 feet 10 inches.
(3.) 6 feet 1 inch, 2 feet 10 inches, 1 foot 8 inches.
(4.) 9 feet 5 inches, 3 feet 5 inches, 2 feet 4 inches.
(5.) 6 feet 3 inches, 5 feet 4 inches, 4 feet 9 inches.
(6.) 10 feet 1 inch, 1 foot 9 inches, 4 feet 3 inches.

2. Find the surface of a cube whose edge is

(1.) 15 inches.
(2.) 3 feet 4 inches.
(3.) 5 feet 10 inches.
(4.) 6 feet 3 inches.
(5.) 5 feet 1 inch.
(6.) 10 feet 3 inches.

3. Find the surface of a cube whose diagonal is 5 feet.

4. Find the edge of a cube which has a total surface of 216 square inches.

5. Find the total surface of a triangular prism, the sides of whose base and height are

(1.) 3 inches, 4 inches, 5 inches, and 4 feet.
(2.) 2 inches, 3 inches, 3 inches, and 1 foot.
(3.) 5 feet, 5 feet, 5 feet, and $\frac{1}{2}$ foot.
(4.) 2 feet 3 inches, 1 foot 5 inches, 1 foot 5 inches, and $1\frac{1}{2}$ foot.
(5.) 2 feet 6 inches, 3 feet 6 inches, and 15 inches.
(6.) 1 foot 9 inches, 1 foot 9 inches, 1 foot 9 inches, and 1 foot 9 inches.

6. What must be the height of a triangular prism, each edge of the base being $2\frac{1}{2}$ inches, so that the total surface is 1 square foot?

7. A triangular prism of 10 inches in height stands on an equilateral base; what must be the edge of the base when the total surface is 3 square feet?

8. Find the lateral surface of a hexagonal prism, each edge of base being 2 feet 3 inches, and height 5 feet.

9. Find the total surface of a cylinder whose radius and height are

(1.) 2 inches, and 1 foot 4 inches.
(2.) 2 feet 3 inches, and 2 feet 3 inches.
(3.) 1 foot 6 inches, and 9 inches.
(4.) 9 inches, and 1 foot 6 inches.
(5.) 1 foot 3 inches, and 3 feet 9 inches.
(6.) 10 feet 6 inches, and 2 feet 4 inches.

10. Find the height of a cylinder whose lateral surface is equal to the surface of the two ends.

11. Find the height, in inches, of a cylinder standing on a base $2\frac{1}{2}$ inches in radius, and whose total surface is 1 square foot.

12. A cylinder 8 inches in height has a total surface of $1\frac{1}{2}$ square feet; find the radius of the base.

13. The lateral surface of a cylinder is $3\frac{1}{3}$ square feet, and the height is 6 inches; find the surface of the ends.

14. The total surface of a cylinder is 20 square feet,

and the height is equal to the diameter of the base; find the area of the base.

15. A cylinder whose height is equal to the radius of the base, has a total surface equal to that of a cube whose edge measures 9 inches; find the height of the cylinder.

16. Find the surface of a 4-inch drain pipe, 1 yard in length.

17. The total surface of a cylinder is 10 square feet, and the height is half the radius; find the height and the area of the two ends.

18. Find the total pressure on a cylindrical vessel measuring 5 feet in diameter, and 15 feet in length, there being 100 lbs. to the square inch.

19. Find the total surface of a square pyramid, the edge of whose base measures 3 inches, and the perpendicular height of the vertex is 5 inches.

20. Find the total surface of a square pyramid, the edge of whose base measures 3 inches, and the slant edge 5 inches.

21. Find the total surface of a square pyramid whose edge measures 3 feet 6 inches, and slant height 5 feet 4 inches.

22. Find the lateral surface of a cylinder whose diameter is 8 feet 3 inches, and length 6 feet 8 inches.

23. Find the total surface of the frustum of a pyramid whose ends are squares measuring 5 and 3 inches in the side, and the slant height is 12 inches.

24. Find the lateral surface of a cone whose diameter is 13 feet 6 inches, and slant side 18 feet 8 inches.

25. Find the lateral surface of a cone whose diameter is 17 feet 2 inches, and altitude 21 feet.

26. Find the lateral surface of a cone whose altitude is $1\frac{1}{2}$ yard, and diameter $\frac{3}{4}$ yard, and the cost of gilding the same at $\frac{1}{4}$d. per square inch, including the base.

27. The sides of a triangle are 3, 4, 5 inches; if it revolve round the hypothenuse, find the whole surface generated by the other two sides.

28. Find the surfaces generated if the same triangle revolves round each of the other sides.

29. If a, b, c be the sides of a right-angled triangle, shew that the area generated by the revolution of the triangle round the hypothenuse is equal to the sum of the areas generated by revolving round the sides.

30. Find the lateral surface of the frustum of a cone whose top and bottom diameters are 5 feet 4 inches and 8 feet 3 inches, the slant height being 10 feet 4 inches.

31. Find the lateral surface of the frustum of a cone whose top and bottom diameters are 15 feet 4 inches, and 26 feet 8 inches, and the depth 23 feet 9 inches.

32. Find the lateral surface of a bucket whose top diameter measures 2 feet 4 inches, and the bottom diameter 1 foot 6 inches, the depth being 1 foot 9 inches.

33. Compare the lateral surfaces of a cone and cylinder standing upon the same base, and having the same vertical height.

34. The lateral surface of a right cone is 650 square inches, and the slant height is 25 inches; find the diameter of the base.

35. The lateral surface of a right cone is 10 square feet, and the radius of the base is 1 foot 4 inches; find the slant height.

36. The lateral surface of a right circular cone is 750 square inches, and the circumference of base is 50 inches; find the vertical height of the cone.

37. How many yards of canvas $\frac{3}{4}$ths of a yard wide will be required for a conical tent 20 feet in diameter, and 15 feet high?

38. Divide a right cone into two parts by a plane parallel to the base, so that the lateral surfaces of the parts are equal.

39. Find the surface of a sphere of which the radius is

(1.) 9 inches.
(2.) 1 foot 3 inches.
(3.) 3 feet 4 inches.
(4.) 2·6 feet.
(5.) $\frac{1}{2}$ yard.
(6.) $3\frac{1}{2}$ feet.

40. What is the radius of a sphere of which the surface is

(1.) 120 square feet.
(2.) 1 square yard.
(3.) 6 square feet, 72 square inches.

41. Find the radius of a sphere which has the same surface as that of a cone 3 inches in radius, and 3 inches high.

42. Compare the surface of a sphere an inch in diameter with that of a cylinder an inch in diameter and an inch high.

43. Find the surface of a segment of a sphere 6 inches in diameter, the height of the segment being 2 inches.

44. In a sphere whose radius is given, find the height of the segment which has $\frac{1}{3}$d of the surface of the sphere.

45. If we divide a semi-circumference into three equal parts, and make it revolve round its diameter, the surface of the zone generated by the middle arc is equal to the sum of the areas of the other two zones.

46. From what height above the surface of the earth will the $\frac{1}{3}$d part of its surface be seen?

47. From what height above the surface of the earth will the $\frac{1}{n}$th part of the surface be seen?

48. What part of the whole surface will be seen from a height of 5 miles?

49. Find the convex surface of a slice 2 feet high, cut from a globe 17 feet in radius.

50. Shew that the convex surface of a segment of a sphere exceeds the area of the base by the area of a circle whose radius is the height of the segment.

51. What is the height of a segment of a sphere whose convex surface is 100 square inches, the radius of the sphere being 3 feet?

52. If the pressure of the atmosphere at the earth's surface be 15 lbs. to the square inch, how much weight of atmosphere does the earth support?

53. Find the radius of a sphere whose surface is equal to the sum of the surfaces of the two spheres whose radii are r and r'.

54. Find the surface of a ring of circular cross section whose outer and inner diameters measure 12 and 10 inches.

55. The outer diameter of a ring of circular cross section $= d$, the thickness of the ring, t; find the surface.

56. Find the total surface of a flat ring whose outer and inner diameters are C and c, and the height h.

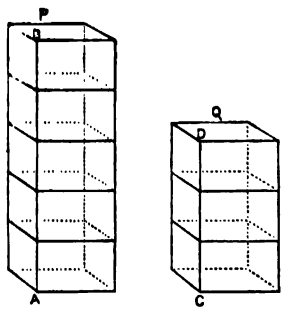
57. Find the total surface of a flat ring whose outer diameter is 10 inches, and inner diameter 8 inches, and height 2 inches.

58. On the radius of a circle OA produced, take a point, B, and through this point draw a tangent, BT, to the circle, shew how to find for what position of B the surface described by BT and the arc AT, when they revolve round OA as axis, may have to each other a given ratio.

III.—MENSURATION OF VOLUMES.

PROP. I.—*Two rectangular parallelopipeds having equal bases are to each other as their altitudes.*

Let P and Q be two rectangular parallelopipeds having equal bases, and let AB and CD be their altitudes.



1. Suppose the altitudes have a common measure,* which is contained say 5 times in AB, and 3 times in CD, so that if AB is divided into 5 equal parts, CD will contain 3 of these parts; then we have

$$\frac{AB}{CD} = \frac{5}{3}.$$

If we now pass planes through the several points of division of AB and CD perpendicular to these lines, the parallelopiped P will be divided into 5 equal parallelopipeds, and Q into 3 parallelopipeds each equal to those in P;

$$\therefore \frac{P}{Q} = \frac{5}{3}.$$

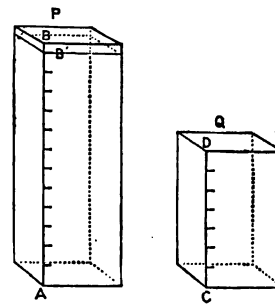
Hence

$$\frac{P}{Q} = \frac{AB}{CD}.$$

2. If the altitudes are incommensurable.

* In finding the area of the rectangle (page 35), we have only considered the case in which each of the sides contains the unit an exact number of times; but it is evident that the general demonstration is similar to that given above for the volume of the rectangular parallelopiped.

Let CD be divided into any number of equal parts, and let one of these parts be applied to AB as often as AB will contain it. Since AB and CD are incommensurable, a certain number of these parts will extend from A to a point B′, leaving a remainder BB′ less than one of the parts. Through B′ pass a plane perpendicular to AB, and denote the parallelopiped whose base is the same as that of P or Q, and whose altitude is AB′ by P′; then, since AB′ and CD are commensurable, we have by the last case,



$$\frac{P'}{Q} = \frac{AB'}{CD}.$$

Now, suppose the number of parts into which CD is divided to be continually increased, the length of each part will become less and less, and the point B′ will approach nearer and nearer to B, and ultimately AB′ will become AB, and P′ will become P;

$$\therefore \frac{P'}{Q} \text{ will become } \frac{P}{Q},$$

and

$$\frac{AB'}{CD} \text{ will become } \frac{AB}{CD}.$$

Since, then, the variables $\frac{P'}{Q}$ and $\frac{AB'}{CD}$ are constantly equal, and approach two limiting values, these limiting values are equal;

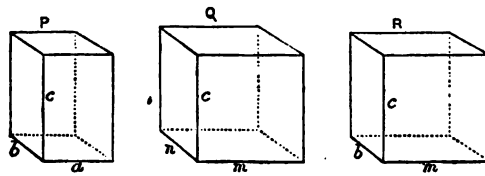
$$\therefore \frac{P}{Q} = \frac{AB}{CD}.$$

Remark.—The three edges of a rectangular parallelopiped which meet at a common vertex being called its dimensions, the preceding theorem may be expressed as follows: *Two rectangular parallelopipeds, which have two*

dimensions in common, are to each other as their third dimensions.

PROP. II. — *Two rectangular parallelopipeds having equal altitudes, are to each other as their bases.*

Let a, b, and c be the three dimensions of P; m, n, and c those of Q, the altitude c being common.



Let R be a third rectangular parallelopiped whose dimensions are m, b, and c; then R has two dimensions, b and c, in common with P, and two dimensions, m and c, in common with Q;

$$\therefore \frac{P}{R} = \frac{a}{m},$$

and

$$\frac{R}{Q} = \frac{b}{n}.$$

Multiplying these ratios, we have

$$\frac{P}{Q} = \frac{ab}{mn}.$$

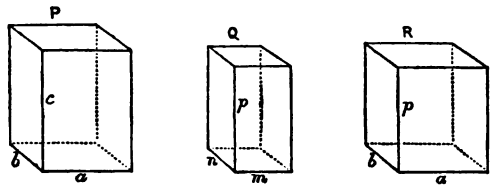
But ab is the area of the base of P, and mn that of Q. Hence P and Q are in the ratio of their bases.

Remark.—We may also express this proposition by saying that: *Two rectangular parallelopipeds, which have one dimension in common, are to each other as the products of the other two dimensions.*

PROP. III.—*Any two rectangular parallelopipeds are to each other as the products of their dimensions.*

Let a, b, and c be the three dimensions of the

rectangular parallelopiped P; m, n, and p those of the rectangular parallelopiped Q.



Let R be a third rectangular parallelopiped whose dimensions are a, b, and p. Then R having two dimensions in common with P, and one in common with Q, we have

$$\frac{P}{R} = \frac{c}{p}.$$

and

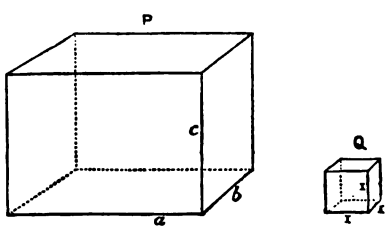
$$\frac{R}{Q} = \frac{ab}{mn}.$$

Multiplying these ratios, we have

$$\frac{P}{Q} = \frac{abc}{mnp}.$$

PROP. IV.—*The volume of a rectangular parallelopiped is equal to the product of its three dimensions, the unit of volume being the cube whose edge is the linear unit.*

Let a, b, and c be the three dimensions of the rectangular parallelopiped P, and let Q be the cube whose edge is the linear unit. By the preceding proposition we have



$$\frac{P}{Q} = \frac{a \times b \times c}{1 \times 1 \times 1} = a \cdot b \cdot c.$$

Now, Q being taken as the unit of measure of volume, $\frac{P}{Q}$ is the numerical measure of P in terms of this unit; therefore the volume of P is equal to the product $a \times b \times c$. Or, taking V to represent the volume, we have

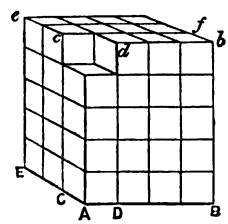
$$V = a \cdot b \cdot c \quad \ldots \quad \ldots \quad (1).$$

Since $abc = ab \times c$, or $ac \times b$, or $bc \times a$, it follows that the volume is equal to the area of one face multiplied by the length of the edge perpendicular to that face.

Let A denote the area of base, h the altitude, we evidently have

$$V = Ah.$$

Remark.—When the three dimensions of the parallelopiped are all exactly divisible by the linear unit, the truth of this proposition is rendered evident by dividing the solid into cubes, each of which is equal to the unit. Let the three edges which meet at the common vertex A be respectively 5, 4, and 3 times the linear unit. It is evident that we may consider the whole solid as being made up of *slices*, AB*bfc*C, and we will have one such slice for every linear unit in AE; that is, 3. Again, each of these slices is composed of *piles*, CAD*dc*, and we evidently have in each slice as many piles as there are units in AB; that is, 4. We therefore have in the whole solid



$$3 \times 4 \text{ piles.}$$

But in every pile there are evidently as many cubes as there are units in Dd; that is, 5.

Hence number of cubes $= 5 \times 4 \times 3$,

or volume $= 5 \times 4 \times 3$.

The more general demonstration given above includes also the cases in which one of the dimensions, or two of

them, or all three of them, are incommensurable with the unit.

Cor. 1.—If the three dimensions of the rectangular parallelopiped are each equal to a, it becomes a *cube*, and we have for its volume

$$V = a^3;$$
$$\therefore a = \sqrt[3]{V}.$$

Cor. 2.—Since (*Euclid*, XI. 31) 'solid parallelopipeds which have equal bases and equal altitudes are equal in volume,' it follows that if V be the volume of any parallelopiped, A the area of its base, and h its height, we have

$$V = Ah;$$
$$\therefore h = \frac{V}{A}, \text{ or } A = \frac{V}{h}.$$

Exercises (8).

1. Find the solid content of a log of wood measuring 25 feet 3 inches in length, 1 foot 9 inches in breadth, and 1 foot 3 inches in depth.

2. Find the number of cubic feet of air in a room 42 feet 8 inches long, 21 feet 4 inches broad, and 18 feet 9 inches high.

3. Find the number of cubic feet and inches in a cube whose side is

(1.) 5 feet 3 inches. | (3.) $3\frac{1}{2}$ yards.
(2.) 15 feet 4 inches. |

4. Find the edge of a cube whose volume is

(1.) 110 cubic inches. | (3.) $1\frac{1}{2}$ cubic yard.
(2.) 2456 cubic feet. |

5. By reducing the side of a cube 2 inches, another cube is formed whose volume is 296 cubic inches less than that of the original cube; find the side of the original cube.

6. Find what length should be cut from a log of wood whose end measures 1 foot 9 inches by 1 foot 3 inches, so that the part cut off will measure $3\frac{1}{2}$ cubic feet.

7. Find the volume of a parallelopiped, the area of whose base and height are

(1.) 15 square feet and 3 feet 5 inches.
(2.) $2\frac{1}{2}$ square yards and 5 feet 9 inches.

8. Find the height of a rectangular parallelopiped whose volume and base are

(1.) 5 cubic feet, 3 square feet.
(2.) $1\frac{1}{2}$ cubic foot, 9 square inches.

9. Find the area of the base of a rectangular parallelopiped whose volume and height are

(1.) 10 cubic feet, 1 foot 2 inches.
(2.) $1\frac{1}{2}$ cubic foot, 9 inches.

10. How many gallons of water will be contained in a cistern in the form of a parallelopiped whose dimensions are

(1.) 6 feet 3 inches long, 2 feet 3 inches broad, and 1 foot 9 inches deep.
(2.) 17 feet 2 inches long, 6 feet 5 inches broad, and 4 feet 3 inches deep.
(3.) 9 feet 9 inches long, 4 feet 6 inches broad, and 3 feet 5 inches deep.

N.B.—*A gallon contains* 277·274 *cubic inches.*

11. Find the weight of water contained in a reservoir which is 12 feet 6 inches long, 9 feet 3 inches deep, and 5 feet 4 inches broad.

N.B.—*A cubic foot of water weighs* $62\frac{1}{2}$ *lbs.*

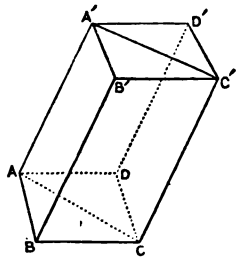
12. A cistern which is 25 feet long and 4 feet wide is filled with water; how far will the surface of the water in the cistern fall if 1000 gallons be drawn off?

13. A reservoir is 12 feet 3 inches in length, and 4 feet 6 inches wide; how many gallons will be run off if the surface sinks $1\frac{1}{2}$ foot?

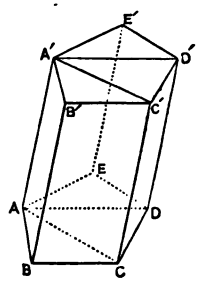
14. Find the weight of a cubic fathom of water.

15. Find the weight of water in a shaft 40 feet deep, and whose section is 3 feet 4 by 3 feet 4.

PROP. V.—*The volume of any prism is equal to the* ·*oduct of its base by its altitude.*



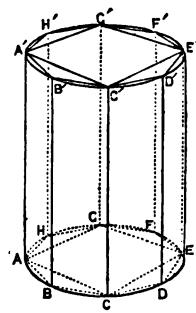
1. Let ABC – A′ be a triangular prism. This prism is equivalent to one-half of the parallelopiped ABCD – A′ constructed upon the edges AB, BC, BB′, and it has the same altitude. The volume of the parallelopiped is equal to its base BD multiplied by its altitude; therefore the volume of the triangular prism is equal to its base ABC, the half of BD, multiplied by its altitude.



2. Let ABCDE – A′ be any prism. It may be divided into triangular prisms by planes passed through a lateral edge AA′ and the several diagonals of the base. The volume of the given prism is the sum of the volumes of the triangular prisms, or the sum of their bases multiplied by their common altitude. But the sum of the bases is the base ABCD; therefore

volume of prism = area of base × altitude.

PROP. VI.—*The volume of a right cylinder on circular base is equal to the product of the base by the altitude.*



Inscribe within the cylinder the right prism ABCD...HA′...H′. The volume of this prism is equal to the area of the base multiplied by the height.

Conceive the number of faces of the inscribed prism to augment continually, then the volume of the prism approaches nearer and nearer to that of the cylinder, and ultimately coincides with it; the area of the base of the prism approaches nearer

and nearer to that of the base of the cylinder, and the height of the prism is always that of the cylinder; hence

volume of cylinder = area of base × height.

Let r denote the radius of the base, and h the height of the cylinder, and V its volume; then

$$V = \pi r^2 h;$$

$$\therefore h = \frac{V}{\pi r^2},$$

$$r = \sqrt{\frac{V}{\pi h}}.$$

Cor. 1.—Similarly, if r' and h' be the radius of the base, and the height of a cylinder of which V′ is the volume, we have

$$V' = \pi r'^2 h',$$

and

$$V = \pi r^2 h;$$

$$\therefore \frac{V}{V'} = \frac{r^2 h}{r'^2 h'} = \frac{r^2}{r'^2} \times \frac{h}{h'}.$$

That is, the volumes of two cylinders *are to each as the products of the squares of their radii by their heights.*

Cor. 2.—If the cylinders are similar, we have

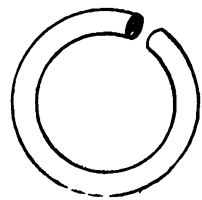
$$\frac{r}{r'} = \frac{h}{h'};$$

$$\therefore \frac{V}{V'} = \frac{r^2}{r'^2} \cdot \frac{r}{r'} = \frac{r^3}{r'^3} = \frac{h^3}{h'^3}.$$

Therefore the volumes of similar cylinders are to each other as the cubes of their radii, or as the cubes of their heights.

PROP. VII.—*To find the volume of a ring.*

1. If the ring be of circular section, we may consider it as a cylinder on a circular base, whose length or height is equal to the mean circumference of the ring, or whose length is equal to that circumference which passes through the centre of every circular section. Let D and d denote the outer and inner diameters of the ring, then we have



$$\frac{D-d}{2} = \text{diameter of circular section;}$$

$$\therefore \pi \cdot \left(\frac{D-d}{4}\right)^2 = \text{area of circular section,}$$

and $$\frac{\pi}{2}(D+d) = \text{mean circumference of ring;}$$

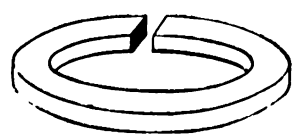
$$\therefore V = \frac{\pi^2}{32}\{D-d\}^2\{D+d\}$$

$$= \frac{(\pi \cdot D - \pi \cdot d)^2(D+d)}{32}$$

$$= \frac{(C-c)^2(D+d)}{32}.$$

Hence the volume of the ring is found *by multiplying the sum of the diameters by the square of the difference of the circumferences, and dividing the product by* 32.

2. If the ring be flat, we may evidently consider it as the difference between two cylinders of the same height.



Let D and d be the outer and inner diameters.

The volume of outer cylinder $= \pi \cdot h\frac{D^2}{4}$,

$$\text{the volume of inner cylinder} = \pi \cdot h\frac{d^2}{4};$$

$$\therefore \text{volume of ring} = \frac{\pi \cdot h}{4}(D^2 - d^2).$$

Hence the volume of the ring is found *by multiplying the difference of the squares of the diameters by one-fourth of the height, and this product by* π. Or *it is equal to the area of the base multiplied by the height.*

Remark.—The expression here given for the volume —namely:

$$V = \frac{\pi h}{4}(D^2 - d^2)$$

may be put in the form:

$$V = h\left(\frac{D-d}{2}\right) \times \pi\left(\frac{D+d}{2}\right);$$

but $h\left(\frac{D-d}{2}\right)$ is evidently the area of the cross section, and $\pi\left(\frac{D+d}{2}\right)$ is the mean circumference; hence, as before, the volume *is equal to the area of the cross section multiplied by the mean circumference.*

Exercises (9).

1. Find the volume of the following prisms whose base and height are

(1.) 3 square feet and 1 foot 9 inches.

(2.) $1\frac{1}{2}$ square yard and 2 feet 4 inches.

2. Find the volume of a triangular prism, the sides of whose base are 17, 20, and 25 inches, and whose height is 48 inches.

3. Find the volume of a prism 30 inches high, on an equilateral base, each of whose sides is 15 inches.

4. Find the height of the prism standing on a triangular base whose sides are 15, 17, and 20 inches, and which contains $2\frac{1}{2}$ cubic feet.

5. Find the volume of a cylinder the radius of whose base and the height are

(1.) 2 feet 3 inches and 3 feet 6 inches.

(2.) 1 foot 9 inches and 1 foot 9 inches.

(3.) 5 feet 6 inches and 10 feet 4 inches.

6. Find the radius of the base of a cylinder whose volume is 10,000 cubic inches, and height 2 feet 6 inches.

7. Find the weight of water contained in a cylindrical boiler, the radius of whose base is 2 feet 3 inches, and height 10 feet.

8. What must be the height of a cylinder, the radius of whose base is 1 foot 9 inches, and whose volume is 12,600 cubic inches.

9. Compare the height of a cylinder whose radius is r with that of a hexagonal prism whose side is s, the volumes of both being alike.

10. Find the number of cubic yards of earth taken from a well whose diameter is 3 feet 6 inches, and whose depth is 30 feet.

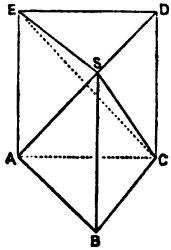
11. Find how many coins, each $\frac{1}{32}$d of an inch thick, and $\frac{3}{8}$ths of an inch in radius, can be made out of a bar of metal 9 inches long, 3 inches broad, and 2 inches thick.

12. Find the quantity of metal in a pipe of 4 inches bore, $\frac{1}{2}$ inch thick, and 10 feet long.

13. What is the ratio of the volumes of two cylinders of which the lateral surfaces are equal?

14. What is the ratio of the lateral surfaces of two cylinders of which the volumes are equal?

PROP. VIII.—*To find the volume of a triangular pyramid.*



Let S – ABC be a triangular pyramid. Through one edge of the base, as AC, pass a plane, ACDE, parallel to the opposite lateral edge, SB, and through S pass a plane, SED, parallel to the base, the prism, ABC – E, has the same altitude as the given pyramid.

Taking away the given pyramid, S – ABC, from the prism, there remains a quadrangular pyramid whose base is the parallelogram, ACDE, and vertex, S. The plane, SEC, passed through SE and SC, divides this pyramid into two triangular pyramids, S – AEC and S – ECD, which are equivalent to each other, since their triangular bases, AEC, ECD, are halves of the parallelogram, ACDE, and their common altitude is the perpendicular from S upon the plane, ACDE. The pyramid, S – ECD, may be regarded as having ESD for its base, and its vertex at C, therefore it is equivalent to the given pyramid, S – ABC; so also will the pyramid, S – EAC, be equal to S – ABC. Therefore the given pyramid is equal to one-third of the whole prism,

or volume of pyramid $= \frac{1}{3}$ volume of prism.

Let A = area of base, h = the height;

$$\therefore V = \tfrac{1}{3}A \cdot h;$$

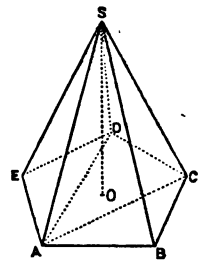
$$\therefore A = \frac{3V}{h};$$

$$h = \frac{3V}{A}.$$

Cor. 1.—Hence it follows that *the volume of any pyramid is equal to one-third of the product of its base by its altitude.*

For any pyramid, S – ABCDE, may be divided into

triangular pyramids by passing planes through an edge, SA, and the diagonals, AD, AC, &c. of its base. The bases of these pyramids are the triangles which compose the base of the given pyramid, and their common altitude is the altitude SO of the given pyramid. The volume of the given pyramid is equal to the sum of the volumes of the triangular pyramids, which is one-third of the sum of their bases multiplied by their common altitude.



Then $$V = \tfrac{1}{3}Ah.$$

Cor. 2.—Hence if A′ and h' be the area of the base, and the height of the pyramid whose volume is V′, we have

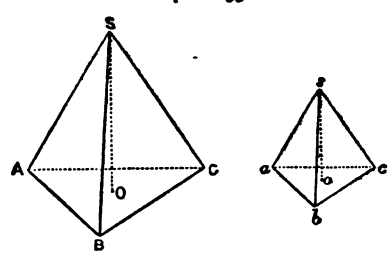
$$V' = \tfrac{1}{3}A'h';$$
$$\therefore \frac{V}{V'} = \frac{Ah}{A'h'}.$$

Hence *the volumes of two pyramids are to each other as the products of the areas of their bases by their heights.*

Cor. 3.—If the pyramids are similar, we have

$$\frac{V}{V'} = \frac{A \cdot h}{A' \cdot h'} = \frac{BC^2}{bC^2} \cdot \frac{h}{h'}.$$

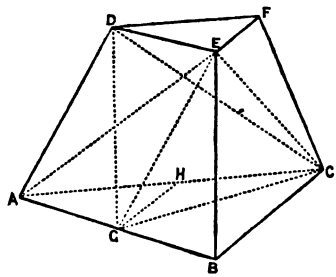
But
$$\frac{h}{h'} = \frac{SB}{Sb} = \frac{BC}{bc}.$$
$$\therefore \frac{V}{V'} = \frac{BC^3}{bc^3}.$$



Hence *the volumes are to each other as the cubes of the edges.*

PROP. IX.—*To find the volume of a frustum of a pyramid.*

1. Consider first the case of the frustum of a triangular pyramid.



Let ABCDEF be the frustum of the triangular pyramid whose volume is required to be found.

Suppose a plane to pass through E and AC, and another through E and DC; these two planes divide the whole frustum into three pyramids—viz.:

1. The pyramid EABC, having the triangle ABC for its base, and the height of the frustum for its altitude.

2. The pyramid EDCF, having its vertex at C, DEF for its base, and the height of the frustum for its altitude.

3. The pyramid EACD, which has its vertex at E, and the triangle ACD for its base.

We will now examine the third pyramid. If we draw through E the line EG parallel to AD, it will also be parallel to the plane ADC. Let the vertex E slide down the line EG to G; the pyramid EACD or Q will be equivalent to the pyramid GACD, for they have the same base and the same altitude. But GACD may be considered as having D for its vertex, and AGC for its base; that is, the pyramid Q has AGC for its base, and for its altitude the height of the frustum.

Through G draw GH parallel to BC, and therefore parallel to EF; then the triangle AGH is evidently equal to the triangle DEF.

Now $\frac{\text{triangle ABC}}{\text{triangle AGC}} = \frac{AB}{AG}$, since they have the same altitude.

But $$\frac{AB}{AG}=\frac{AC}{AH}=\frac{\text{triangle AGC}}{\text{triangle AHG}};$$

$$\therefore \frac{\text{triangle ABC}}{\text{triangle AGC}}=\frac{\text{triangle AGC}}{\text{triangle AHG}}.$$

But triangle AHG = triangle DEF;

$$\therefore \frac{\text{triangle ABC}}{\text{triangle AGC}}=\frac{\text{triangle AGC}}{\text{triangle DEF}};$$

$\therefore$ triangle AGC is a mean proportional between the lower and upper bases.

Hence *the frustum of the pyramid is made up of three pyramids, having for their common altitude the height of the frustum,* and for bases—*the first, the lower base of the frustum; the second, the upper base; and the third, a mean proportional between the two bases.*

Let A and a denote the areas of the lower and upper bases, h the height of the frustum; we have for the volume,

$$V=\frac{h}{3}\{A+a+\sqrt{Aa}\}.$$

Remark.—Sometimes, in place of having given the two bases A and a, we have given one of the bases as A and the ratio $\frac{s}{S}$ of two homologous sides of the two bases; we then have

$$\frac{a}{A}=\frac{s^2}{S^2};$$

$$\therefore a=\frac{s^2}{S^2}A.$$

We then have the volume,

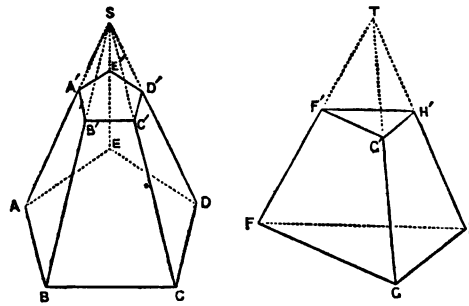
$$V=\frac{h}{3}\left\{A+\frac{s^2}{S^2}A+\sqrt{A\cdot\frac{s^2}{S^2}A}\right\}$$

$$=\frac{h}{3}\left\{A+\frac{s^2}{S^2}A+\frac{s}{S}A\right\}$$

$$=\frac{Ah}{3}\left\{1+\frac{s}{S}+\frac{s^2}{S^2}\right\}.$$

2. Consider next the frustum of any pyramid.

Let SABCDE be any pyramid whatever. Conceive a triangular pyramid, TFGH, having for base a triangle, FGH, equal to the polygon, ABCDE, and of the same



height; this triangular pyramid will have the same volume as that of the proposed pyramid. If we suppose the two bases placed upon the same plane, and if we cut the two pyramids by the same plane parallel to the plane of the bases, this plane will evidently determine in the two pyramids equivalent sections, A'B'C'D'E' and F'G'H'. The upper portions of the two pyramids, having the same height and equal bases, must be equal, and if these portions be removed, the frustum of the proposed pyramid and that of the triangular pyramid are equal. But the frustum of the triangular pyramid is equal to the sum of three pyramids, having for their common height the height of the frustum, and for bases—the first, the lower base of the pyramid; the second, the upper base; the third, a mean proportional between these bases. It is the same, therefore, for the frustum of the polygonal pyramid. We have then, as before,

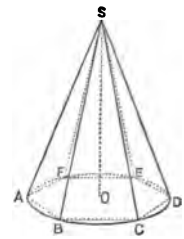
$$V = \frac{h}{3}\{A + a + \sqrt{Aa}\}.$$

Remark.—If $a = o$, the frustum becomes a complete pyramid, and $V = \frac{h}{3}A$, as was to be expected.

If $a = A$, the frustum becomes a prism, and $A = \frac{h}{3} \times 3A = Ah$, as was to be expected.

PROP. X.—*To find the volume of a cone.*

Inscribe within the cone, SAD, the regular pyramid, SABCDEF; the volume of this pyramid *is measured by the area of its base multiplied by* $\frac{1}{3}d$ *of its height*, SO. If now we double indefinitely the number of the faces of the pyramid, the volume of the pyramid approaches nearer and nearer to the volume of the cone, and ultimately coincides with it; the base of the polygon approaches nearer and nearer to the base of the cone, and ultimately coincides with it. The height of the pyramid is always the height of the cone, therefore the volume of a right cone on circular base *is measured by the area of the base multiplied by a third of the height.*



Let r = radius of base, and h = height of cone, then we have the volume

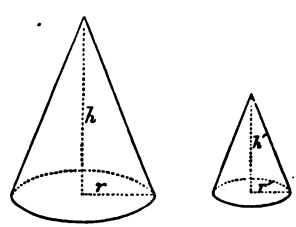
$$V = \pi r^2 \times \frac{h}{3} = \frac{1}{3}\pi r^2 h.$$

If V' = the volume of another cone whose height is h', and radius of base r', we have

$$V' = \tfrac{1}{3}\pi r'^2 h';$$

$$\therefore \frac{V}{V'} = \frac{r^2 h}{r'^2 h'}.$$

Hence *the volumes of two cones are to each other as the products of the squares of their bases by their heights.*



If the cones are similar, we have

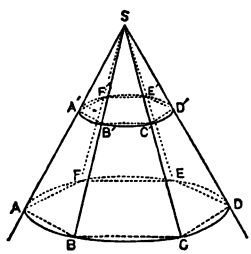
$$\frac{r}{r'}=\frac{h}{h'};$$

$$\therefore \frac{V}{V'}=\frac{h^3}{h'^3}=\frac{r^3}{r'^3}.$$

Hence *the volumes of similar cones are to each other as the cubes of their radii, or of their heights.*

PROP. XI.—*To find the volume of the frustum of a cone.*

Consider the frustum of the regular pyramid inscribed in the frustum of the cone. The volumes of the two pyramids of which this frustum is the difference, have, when the number of sides is indefinitely increased, as their ultimate values, the volumes of the two cones, whose difference is the proposed frustum. We will then have the volume of the frustum of the cone by taking the ultimate value of the volume of the frustum of the pyramid.



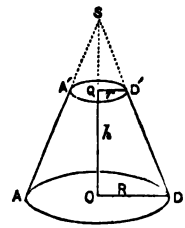
Let V, A, a, h be the volume, the bases, and the height of the frustum of the cone, and V', A', a' the volume and bases of the inscribed frustum of the pyramid, we have

$$V'=\frac{h}{3}\{A'+a'+\sqrt{A'a'}\}.$$

But when the number of faces of the pyramid is indefinitely increased, V' tends towards V, and ultimately coincides with it; A' tends towards A, and ultimately coincides with it; and a' with a; therefore, for the volume of the frustum of the cone,

$$V = \frac{h}{3}\{A + a + \sqrt{Aa}\}.$$

If R is the radius of the lower, and r of the upper base, we have



$$V = \frac{h}{3}\{\pi R^2 + \pi r^2 + \pi Rr\}$$

$$= \frac{\pi h}{3}\{R^2 + r^2 + Rr\}$$

$$= \frac{h}{3}(\pi R^2) + \frac{h}{3}(\pi r^2) + \frac{h}{3}(\pi Rr).$$

Remark 1.—Hence the frustum of the cone is equivalent to three cones, having for their common altitude the height of the frustum, and for bases—the first, the lower base of the frustum; the second, the upper base; and the third, a mean proportional between the two bases.

We might have obtained the volume of the frustum of the cone very readily by taking the difference between the volume of the complete cone SAD, and that of the cone SA′D′. We must in this case first find the heights of these two cones—namely, SO and SQ.

Let $SQ = x$; $\therefore SO = x + h$.

Now $x + h : R :: x : r$;

$$\therefore Rx = rx + rh;$$

$$\therefore x = \frac{r}{R - r}h = SQ;$$

$$\therefore \frac{R}{R - r}h = SO;$$

$$\therefore \text{volume of complete cone} = \frac{1}{3} \cdot \frac{Rh}{R - r} \times \pi R^2,$$

$$\text{volume of small cone} = \frac{1}{3} \cdot \frac{rh}{R - r} \times \pi r^2;$$

$$\therefore \text{volume of frustum} = \frac{h\pi}{3(R - r)}\{R^3 - r^3\},$$

or $$V = \frac{h\pi}{3}\{R^2 + r^2 + Rr\}$$

$$= \frac{h}{3}\{\pi R^2 + \pi r^2 + \pi Rr\}$$

$$= \frac{h}{3}\{A + a + \sqrt{Aa}\}, \text{ as before.}$$

Remark 2.—In the formula,

$$V = \frac{h\pi}{3}(R^2 + r^2 + Rr).$$

Since $R^2 + r^2 + Rr = 3\left(\frac{R+r}{2}\right)^2 + \left(\frac{R-r}{2}\right)^2$.

We can put the expression for the volume under the following form:

$$V = \pi\left(\frac{R+r}{2}\right)^2 \cdot h + \tfrac{1}{3}\pi\left(\frac{R-r}{2}\right)^2 \cdot h.$$

We see, then, that V is equal to the sum of the volumes of a cylinder and of a right cone on circular base, which have the same height as the frustum of the given cone, and such that the radii of the bases are respectively

$$\frac{R+r}{2} \text{ and } \frac{R-r}{2}.$$

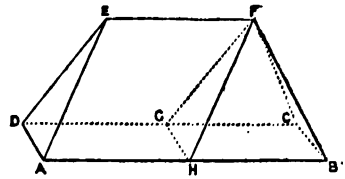
If the difference between R and r is very small, the volume of the cone may be neglected, and we have for an approximate value of the volume,

$$V = \pi\left(\frac{R+r}{2}\right)h.$$

This formula is frequently employed in practice, but it should be borne in mind that it is not exact, and cannot be employed with any degree of accuracy unless $R - r$ is very small.

Prop. XII.—*To find the volume of a wedge.*

Let the base be rectangular. Through F pass a plane parallel to the plane EDA. This plane will divide the wedge into two parts—namely, AHFGDE, and the pyramid FHBC.



Let EF, the edge of the wedge, $= e$,
AB, the length of the base, $= a$,
AD, the breadth of the base, $= b$.
and h the height.

$$\text{Volume of AHFGDE} = e \cdot b \cdot \frac{h}{2},$$

$$\text{volume of FHBCG} = (a-e)b \cdot \frac{h}{3};$$

$\therefore$ total volume, or volume of wedge,

$$\begin{aligned} V &= be\frac{h}{2} + (a-e)b \cdot \frac{h}{3} \\ &= \tfrac{1}{6}\{3beh + 2bh(a-e)\} \\ &= \tfrac{1}{6}\{beh + 2abh\} \\ &= \frac{bh}{6}\{2a+e\}. \end{aligned}$$

Interpreting this, we have the following rule for finding the volume of the wedge:

Add the length of the edge to twice the length of the base, multiply the sum by the width of the base, and the product by the height of the wedge; one-sixth of the result will be the volume of the wedge.

Exercises (10).

1. A triangular pyramid on equilateral base, each of whose sides is 15 feet, has an altitude of 30 feet; find its volume.

2. The base of a pyramid is a regular hexagon whose side is 16 feet, and whose height is 25 feet; find its content.

3. Find the volume of a pyramid on a square base whose side is 120 feet, and whose slant edge is 120 feet.

4. Find the volume of a cone, the radius of whose base is 4 feet 6 inches, and whose height is equal to its circumference.

5. Find the volume of the frustum of a cone, the circumferences of whose bases are 66 and 56 feet, and whose height is 4 feet.

6. Find the solid content of the frustum of a cone, the perpendicular height of which is 7 feet, and the radii of the ends are 5 feet and 4 feet.

7. Find the weight of water in a tank in the form of the frustum of a cone, the radii of the ends being 10 feet and 8 feet, and the distance between being 5 feet.

8. A cylinder and the frustum of a cone have the same height, and stand on the same base; find the ratio of the radii of the ends of the frustum when its volume is one-half the volume of the cylinder.

9. How often can a wine-glass in the form of a cone, $2\frac{1}{2}$ inches in diameter and 2 inches deep, be filled out of a cylindrical tumbler, 4 inches in diameter and $3\frac{1}{2}$ inches deep?

10. Shew that the difference between the volume of the frustum of a pyramid and the volume of a prism of the same altitude, whose base is a section of the frustum parallel to its bases, and equidistant from them, is

$$\frac{h}{12}(\sqrt{A} - \sqrt{a})^2,$$

if A and a denote the areas of the base, and h the height of the frustum.

11. Find the volume of a wedge, the edge of which is $3\frac{1}{2}$ inches, and the length and breadth of the base 5 inches and 2 inches, and the height 10 inches.

12. The length and breadth of the base of a wedge are 10 inches and 5 inches, the height 1 foot, and the edge 1 foot; find the volume.

13. Calculate the lateral surface, the total surface, and the volume of a cone as a function of the slant height, the radius of the base being equal to the altitude.

14. Shew how to divide the lateral surface of a cone into n equal parts by planes parallel to the base.

PROP. XIII.—*Lemma. If a triangle revolves round an axis situated in its plane, and passing through the vertex without crossing its surface, the volume generated is equal to the surface generated by the base multiplied by one-third of the altitude.*

Let ABC be the triangle revolving round the axis XY though the vertex A, then the volume generated is equal

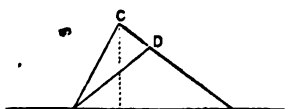


Fig. 1.

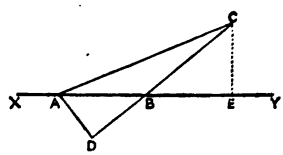


Fig. 2.

to the area generated by the base BC multiplied by one-third of the altitude AD.

We distinguish three cases.

1. When one of the sides of the triangle, as AB, lies along the axis (figs. 1 and 2).

Draw CE perpendicular to the axis. According as this perpendicular falls within the triangle (fig. 1), or without it (fig. 2), the volume generated is the sum or difference of the cones generated by the right-angled triangles, ACE and BCE. We have (III. X.)

Volume of cone generated by ACE $= \frac{1}{3}\pi \cdot CE^2 \cdot AE$,

volume of cone generated by BCE $= \frac{1}{3}\pi \cdot CE^2 \cdot BE$.

If we take their sum, we have in fig. 1, $AE + BE = AB$; and their difference we have in fig. 2, $AE - BE = AB$. Therefore, in either case,

Volume generated by ABC $= \frac{1}{3}\pi \times CE^2 \times AB$

$= \frac{1}{3}\pi \cdot CE \cdot CE \cdot AB$.

But $CE \cdot AB = CB \cdot AD$;

$\therefore$ volume by $ABC = \frac{1}{3}\pi \cdot CE \cdot CB \cdot AD$

$$= \pi \cdot CE \cdot CB \cdot \frac{AD}{3}.$$

But $\pi \cdot CE \cdot CB$ measures the surface generated by CB;

$$\therefore \text{volume by } ABC = \text{surface by } BC \times \frac{AD}{3}.$$

2. When the triangle has only the vertex in the axis, and the base BC, when produced, meets the axis in F (fig. 3).

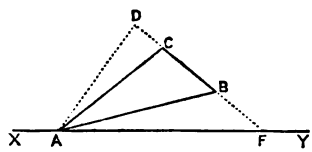


Fig. 3.

The volume generated is then the difference of the volumes generated by the triangles ACF and ABF; but by the last case,

$$\text{Volume generated by } ACF = \text{area by } CF \times \frac{AD}{3},$$

$$\text{volume generated by } ABF = \text{area by } BF \times \frac{AD}{3};$$

$\therefore$ volume generated by $ABC = (\text{area by } CF - \text{area by } BF) \times \frac{AD}{3}$

$$= \text{area by } CB \times \frac{AD}{3}.$$

3. When the triangle has only the vertex A in the axis, and the line BC is parallel to the axis (figs. 4 and 5).

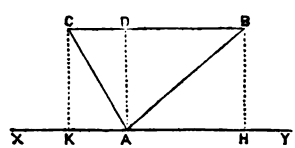


Fig. 4.

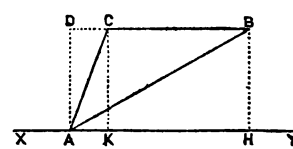


Fig. 5.

The volume generated is the sum (fig. 4) or the

difference (fig. 5) of the volumes generated by the triangles ADB, ACD.

Draw BH and CK perpendicular to the axis.

The volume generated by the triangle ABD is the difference of the volumes of the cylinder by AHBD, and the cone by ABH;

$\therefore$ volume generated by triangle ABD

$$= \pi \cdot AD^2 \cdot BD - \tfrac{1}{3}\pi \cdot AD^2 \cdot BD$$
$$= \tfrac{2}{3}\pi \cdot AD^2 \cdot BD$$
$$= 2\pi \cdot AD \cdot BD \cdot \frac{AD}{3}.$$

But $2\pi \cdot AD \cdot BD$ = area generated by BD;

$\therefore$ volume generated by triangle ABD

$$= \text{area by BD} \times \frac{AD}{3}.$$

Similarly, volume generated by triangle ACD

$$= \text{area by CD} \times \frac{AD}{3}.$$

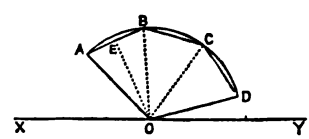
Taking the sum of these (fig. 4), or the difference (fig. 5), we have

$$\text{Volume generated by ABC} = \text{area by BC} \times \frac{AD}{3}.$$

Therefore, in all cases, the *volume generated by the triangle is equal to the area generated by its base multiplied by one-third of the altitude.*

THE SPHERE.

PROP. XIV.—*The volume generated by the sector,* OABCD, *of a regular polygon, turning round an axis,* XY, *drawn in its plane, and passing through its centre, is measured by the area generated by* (AB + BC + CD) *multiplied by* $\frac{OE}{3}$.



Join the centre with B and C; we thus divide the sector into a number of isosceles triangles.

Then, by the last proposition,

Volume generated by triangle OAB

$$= \text{area by AB} \times \frac{OE}{3},$$

volume generated by triangle OBC

$$= \text{area by BC} \times \frac{OE}{3},$$

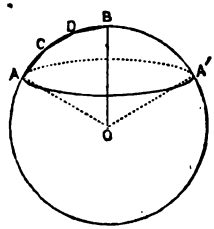
volume generated by triangle OCD

$$= \text{area by CD} \times \frac{OE}{3},$$

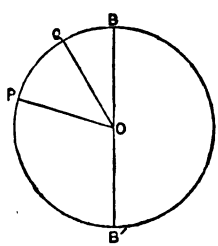
$\therefore$ volume generated by sector OABCD

$$= \text{area by (AB + BC + CD)} \times \frac{OE}{3}.$$

PROP. XV.—*The volume generated by a sector*, AOB, *of a circle turning round one of its sides*, OB, *is measured by the area which is generated by the arc*, AB, *multiplied by one-third of the radius of the circle.*

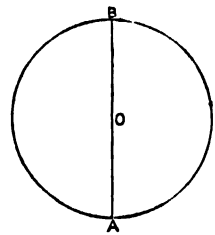


This follows at once from the preceding proposition, since the arc, AB, is the limit towards which tends the sum of the equal chords, AC, CD, DB, when their number is indefinitely increased; and therefore the area generated by the sum of these chords will ultimately be the area generated by the arc, and the perpendicular from the centre will become the radius of the circle.



Remark.—Similarly, it might be shewn that the volume generated by the sector, POQ, turning round any diameter, BOB', is equal to the area generated by the arc, PQ, multiplied by one-third of the radius.

PROP. XVI.—*The volume of a sphere is equal to the product of its surface multiplied by one-third of the radius.*



This follows from the preceding proposition, for if the arc, BA, becomes a semicircle, the surface generated by this arc will be that of the whole sphere; hence the volume of the surface generated—that is, the volume of the sphere—*is equal to its surface multiplied by one-third the radius.*

If V denotes the volume of a sphere whose radius is r, or diameter d, and S its surface, we have

$$V = S \cdot \frac{r}{3} = 4\pi r^2 \cdot \frac{r}{3}$$

$$= \frac{4\pi r^3}{3} = \frac{\pi d^3}{6}.$$

Hence, to find the volume of a sphere, we have the following rule: *Multiply four times the cube of the radius by* $3\frac{1}{7}$, *and divide the product by* 3.

Or *multiply the cube of the diameter by* $3\frac{1}{7}$, *and divide by* 6.

Similarly, if r' be the radius of a sphere whose volume is V', we have

$$V' = \frac{4\pi r'^3}{3},$$

and

$$V = \frac{4\pi r^3}{3};$$

$$\therefore \frac{V}{V'} = \frac{r^3}{r'^3} = \frac{d^3}{d'^3} = \frac{c^3}{c'^3}.$$

Hence *the volumes of spheres are to each other as the cubes of their radii.*

Since

$$V = \frac{4\pi r^3}{3};$$

$$\therefore r = \sqrt[3]{\frac{3V}{4\pi}}.$$

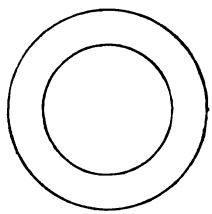
Hence, having given the volume, we can find the radius.

Cor. 1.—The volume of the cylinder circumscribing the sphere whose radius is r, is

$$2\pi r^3;$$

$$\therefore \text{ vol. of sphere : vol. of cylinder :: } \frac{4\pi r^3}{3} : 2\pi r^3 :: 2 : 3;$$

$\therefore$ volume of sphere $= \frac{2}{3}$ volume of circular cylinder.



Cor. 2.—Hence we can find the volume of a spherical shell.

Let D and d denote the outer and inner diameters.

We evidently have

$$\text{Volume of shell} = \frac{\pi D^3}{6} - \frac{\pi d^3}{6}$$

$$= \frac{\pi}{6}\{D^3 - d^3\} = \frac{\pi}{6}\{D - d\}\{D^2 + Dd + d^2\}$$

$$= \frac{\pi}{3} \cdot \left(\frac{D-d}{2}\right)\{D^2 + d^2 + Dd\},$$

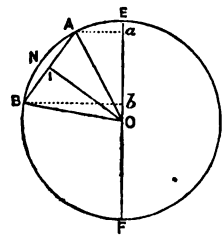
which shews that the volume of the shell *is equal to that of the frustum of a cone, the radii of whose bases are D and d, and whose height is the thickness of the shell.*

Remark.—The volume of the sphere may also be found very readily, by supposing the surface of the sphere composed of an infinite number of small faces, which may be considered as planes, and consequently the volume may be considered as the sum of an infinite number of pyramids having for bases these faces, for the common vertex the centre of the sphere, and for height the radius. But the volume of a pyramid is found (III. VIII.) by multiplying the area of the base by one-third of the height; then the sum of all these pyramids, or the volume of the sphere, will be equal to the sum of the areas of the small faces—that

is, the surface of the sphere—multiplied by one-third of the radius.

Hence, as before, $V = S \cdot \frac{r}{3} = 4\pi r^2 \cdot \frac{r}{3} = \frac{4}{3}\pi r^3$.

PROP. XVII.—*To find the volume generated by a circular segment revolving round a diameter exterior to it.*



Let ANBIA be the circular segment revolving round the diameter, EF, and *ab* the projection of the chord, AB, on the axis.

The volume generated by the segment is evidently the difference between the volume generated by the sector, ABO, and that by the triangle, ABO.

Volume generated by sector AOB

$$= \text{surface by arc AB} \times \frac{OA}{3} = \tfrac{2}{3}\pi \cdot OA^2 \cdot ab,$$

volume generated by triangle AOB

$$= \text{surface by AB} \times \frac{OI}{3} = \tfrac{2}{3}\pi \cdot OI^2 \cdot ab,$$

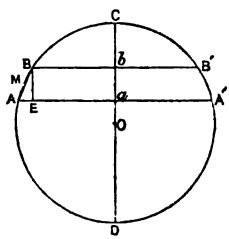
the difference of which gives

$$\text{Volume by segment ANB} = \tfrac{2}{3}\pi \cdot ab(OA^2 - OI^2)$$
$$= \tfrac{2}{3}\pi \cdot ab \cdot AI^2 = \tfrac{2}{3}\pi \cdot ab \cdot \frac{AB^2}{4}$$
$$= \tfrac{1}{6}\pi \cdot ab \cdot AB^2;$$

which shews that the volume is *equivalent to one-sixth of the cylinder whose radius is the chord of the segment, and whose altitude is the projection of the chord on the axis.*

PROP. XVIII.—*The volume of a spherical segment is equal to half the sum of the areas of its bases multiplied by its altitude, together with the volume of a sphere of which that altitude is the diameter.*

Def.—If two parallel planes intersect a sphere, that portion of the sphere between the planes is termed a spherical segment.



Let AA′ and BB′ be the diameters of the two bases, and ab the height of the segment.

Let CD be that diameter of the sphere which is perpendicular to the plane of the bases. The volume, V, will be generated by the revolution of the surface, aAMBb, about the diameter; but this volume is evidently composed of the volume generated by the trapezoid, aABb, and of that by the segment, AMB;

$$\therefore V = \text{volume by } a\text{AB}b + \text{volume by AMB}.$$

But volume generated by aABb is the volume of a frustum of a cone, and is equal to

$$\tfrac{1}{3}\pi \cdot ab\{(\text{A}a)^2 + (\text{B}b)^2 + \text{A}a \cdot \text{B}b\},$$

and volume generated by the segment AMB

$$= \tfrac{1}{6}\pi(\text{AB})^2(ab);$$

$$\therefore V = \tfrac{1}{3}\pi \cdot ab\{(\text{A}a)^2 + (\text{B}b)^2 + \text{A}a \cdot \text{B}b\} + \tfrac{1}{6}\pi(\text{AB})^2(ab);$$

$$\therefore V = \tfrac{1}{6} \cdot \pi \cdot ab\{2(\text{A}a)^2 + 2(\text{B}b)^2 + 2\text{A}a \cdot \text{B}b + \text{AB}^2\} \ldots (1).$$

Draw BE perpendicular to Aa.

Then $$\text{AB}^2 = \text{BE}^2 + \text{AE}^2,$$

and $$\text{AE}^2 = (\text{A}a - \text{B}b)^2$$

$$= (\text{A}a)^2 + (\text{B}b)^2 - 2\text{A}a \cdot \text{B}b;$$

$$\therefore \text{AB}^2 = (ab)^2 + (\text{A}a)^2 + (\text{B}b)^2 - 2\text{A}a \cdot \text{B}b.$$

Substituting this value of AB^2 in (1), we have

$$V = \tfrac{1}{6}\pi \cdot ab\{3(\text{A}a)^2 + 3(\text{B}b)^2 + (ab)^2\}$$

$$= \frac{\pi}{2}\{(\text{A}a)^2 + (\text{B}b)^2\}(ab) + \frac{\pi}{6}(ab)^3$$

$$= \left\{\frac{\pi(\text{A}a)^2 + \pi(\text{B}b)^2}{2}\right\}(ab) + \frac{\pi}{6}(ab)^3 \quad \ldots \quad (2).$$

Denoting the radii of the bases by R and r, and the height of the segment by h, we have for the volume,

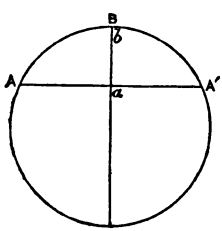
$$V = \tfrac{1}{2}(\pi R^2 + \pi r^2)h + \frac{\pi h^3}{6}.$$

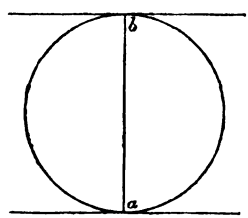
The first part of the right-hand expression denotes *half the sum of the areas of the bases multiplied by the height of the segment;* and the last part of the expression denotes *the volume of a sphere whose diameter is the height of the segment;* or we may say the first part of the expression is that of *a cylinder whose base is half the sum of the given bases, and altitude equal to the height of the segment.*

Cor. 1.—If one of the bounding planes—the upper one—touch the sphere, then this base is reduced to a point. We then have for the volume

$$V = \frac{\pi}{2}(Aa)^2 \cdot ab + \tfrac{1}{6}\pi(ab)^3$$

$$= \frac{\pi}{2}R^2 \cdot h + \tfrac{1}{6}\pi h^3.$$





If both of the bounding planes touch the sphere, then both bases are reduced to points;

$$\therefore V = \tfrac{1}{6}\pi(ab)^3 = \frac{\pi}{6}D^3 = \text{volume of the whole sphere,}$$

as might have been expected, since the spherical segment under these conditions becomes the whole sphere.

PROP. XIX.—*To measure the length of the radius of a solid sphere.*

We here suppose that only measurements on the surface are possible.

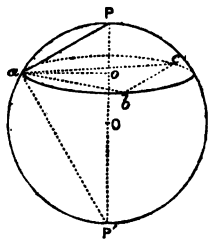


Fig. 1.

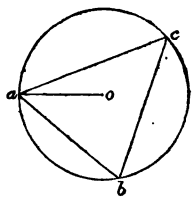


Fig. 2.

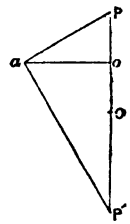


Fig. 3.

1. With any point, P (fig. 1), of the given surface as a pole, and with any arbitrary opening of the compasses, describe a circumference, *abc*, on the surface. The rectilinear distance, P*a*, being the arbitrary opening of the compasses, is a known line; take any three points, *abc*, in this circumference, and with the compasses measure the rectilinear distances, *ab*, *bc*, *ca*.

2. On the plane of the paper construct a triangle, *abc* (fig. 2), with sides equal to *ab*, *bc*, *ca*, and find the centre, O, of the circle circumscribing this triangle. The radius, *ao*, of this circle is the radius of the circle, *abc*, of fig. 1.

3. With the radius, *ao*, as a side, and the known distance, P*a*, as the hypothenuse, construct a right-angled triangle, *ao*P (fig. 3). Draw AP′ perpendicular to *a*P, meeting P*o* produced in P′. Then it is evident that PP′ thus determined is equal to the diameter of the given sphere, and its half, P*o*, is the required radius.

Exercises (11).

1. Find the volume of a sphere whose radius is

(1.) 5 inches. | (3.) 3 feet.
(2.) 2 feet 6 inches. |

2. Find the radius of a sphere whose volume is

(1.) 5000 cubic inches. | (2.) 3 cubic feet.

3. Find the volume of a sphere whose surface is

(1.) 25 square feet. | (2.) 100 square inches.

4. Find the diameter of a sphere which has the same volume as a cylinder whose height $= h$, and radius of base $= r$.

5. Find the radius of a sphere which has the same volume as a cone whose height $= h$, and radius of base $= r$.

6. Find the quantity of metal in a spherical shell whose outer diameter $= d$ inches, and whose thickness $= t$ inches.

7. A cone circumscribes a given sphere, and the height of the cone is twice the diameter of the sphere; shew that the total surface of the cone is twice that of the sphere, and the volume of the cone twice that of the sphere.

8. Prove that when the side of the frustum of a cone is equal to the sum of the radii of the bases, the height of the frustum is twice the geometric mean of the radii, and the volume is found by multiplying the total surface by the sixth part of the height.

9. A triangle whose sides are 5, 4, and 3 revolves round the largest side as axis; compare the volumes generated by the other two sides.

10. Find the volume generated by an equilateral triangle revolving round one of its sides.

11. The volumes generated by a parallelogram turning successively round two adjacent sides are inversely as the lengths of the sides.

12. Find the volume generated by a regular hexagon revolving round one of its sides.

13. In a sphere of which the radius is R, and centre

O, to draw a plane AB cutting the sphere, such that the ratio of the surface of the zone BPA to the surface of the cone BOA, which has for vertex the centre O of the sphere, and for base the circle AB, may be $\frac{1}{4}$.

14. Divide a sphere of given radius into two parts, so that the volume of the one part may be to that of the other in the ratio of $m : n$.

15. Express the volume of a sphere as a function of the circumference of a great circle.

16. Prove that the volume of a cylinder circumscribing a sphere is a mean proportional between the volume of the sphere, and that of the equilateral cone circumscribing. The cone is said to be equilateral when the diameter of the base is equal to the slant height.

17. The diameters of the earth, of the moon, and of the sun being proportional to the numbers

$$1, \frac{3}{11}, 112;$$

how are the volumes of the moon and of the sun expressed if that of the earth is taken for unity?

18. If we join the middle points of two sides of a triangle by a straight line, and make the triangle revolve round the remaining side; shew how to find the volumes generated by each of the two parts of the triangle.

19. The volumes generated by a rectangle turning successively round two adjacent sides are v and v'; find the diagonal of this rectangle.

20. Having given a sphere of which the radius is R, construct a cone which has the same volume as the sphere, and of which the height is half the radius of the sphere; what will the radius of the base be?

21. AB is the diameter of a sphere; shew how to draw a plane perpendicular to this diameter, dividing the sphere into two parts, whose surfaces are to each other in the ratio of 2 to 3.

22. An isosceles triangle of which the base is b, and other sides each equal to a, turns round its base; express the volume thus generated.

23. An equilateral triangle, ABC, revolves first round the side BC, and afterwards round a line DE, through A, parallel to BC; compare the volumes thus generated.

24. The difference of the radii of two spheres $= d$, and the difference of their volumes $= V$; find the radius of each.

25. The volumes of a right circular cone, of a sphere, and of a right cylinder of the same height, are proportional to the numbers 1, 2, 3, when the cone and the cylinder have for bases a great circle of the sphere.

MISCELLANEOUS EXERCISES.

1. *Given the total surface of a right circular cone, and also its volume, to determine the cone.*

Let $x =$ the radius of the base,

$y =$ slant height,

$4\pi a^2 =$ total surface of cone,

$\frac{4\pi b^3}{3} =$ volume of cone.

We evidently have the two equations,

$$x^2 + xy = 4a^2 \quad \ldots \quad \ldots \quad (1),$$

$$x^2\sqrt{y^2 - x^2} = 4b^3 \quad \ldots \quad \ldots \quad (2).$$

Substituting in (2) for y, the value,

$$y = \frac{4a^2 - x^2}{x} \quad \ldots \quad \ldots \quad (3);$$

found from (1), we have

$$ax\sqrt{4a^2 - 2x^2} = 2b^3 \quad \ldots \quad (4);$$

or squaring both sides,

$$a^2x^4 - 2a^4x^2 + 2b^6 = 0 \quad \ldots \quad (5).$$

This equation has two roots, *real* and *positive*

$$\text{if } b^6 < \frac{a^6}{2};$$

from (4), we see that each of these roots must be less

than $a\sqrt{2}$, and consequently the values of y will both be positive and greater than those of x. We have therefore *two* solutions.

If $$b^6 = \frac{a^6}{2}, \quad x = a, \text{ and } y = 3a;$$

and there is only one solution.

2. *In a given sphere, it is required to inscribe a cylinder whose total surface is given.*

Let r = the radius of the sphere,
x = the radius of the base of the cylinder,
$2y$ = the height of the cylinder,
$4\pi a^2$ = total surface of cylinder.

We will evidently have the following equations to determine x and y.

$$x^2 + y^2 = r^2 \quad \ldots \quad \ldots \quad \ldots \quad (1),$$
$$2\pi x^2 + 4\pi xy = 4\pi a^2 \quad \ldots \quad \ldots \quad (2).$$

If we substitute in (1), the value of

$$y = \frac{2a^2 - x^2}{2x} \quad \ldots \quad \ldots \quad (3);$$

found from (2), we have

$$5x^4 - 4(a^2 + r^2)x^2 + 4a^4 = 0 \quad \ldots \quad (4);$$

$$\therefore x = \sqrt{\frac{2(a^2 + r^2) \pm 2\sqrt{(a^2 + r^2)^2 - 5a^4}}{5}} \quad (5).$$

As x is essentially positive, it is unnecessary to prefix the outside radical with the double sign $\pm$.

It follows from the above, that in order that the values of x^2 in (4) may be real, we must have

$$(a^2 + r^2)^2 \geq 5a^4,$$

or $$a^2 + r^2 \geq a^2\sqrt{5};$$

or, which is the same thing, we must have the condition

$$a^2 \angle \frac{r^2}{\sqrt{5}-1},$$

or
$$a^2 \angle \frac{r^2(\sqrt{5}+1)}{4},$$

or
$$4\pi a^2 \angle \pi r^2(\sqrt{5}+1);$$

that is, the total surface of the given cylinder must be less than

$$\pi r^2(\sqrt{5}+1).$$

If this condition is fulfilled, the two values of x^2 are real and positive.

Substituting each of these values of x^2 in (3), we have corresponding values of y. But as the value of y ought also to be positive, we must have

$$x^2 \angle 2a^2.$$

3. In a triangle ABC, draw AD perpendicular to BC; from D draw DE perpendicular to AB, and DF perpendicular to AC. Having given the lengths of these perpendiculars equal respectively to P, p_1, and p_2, find the sides of the triangle.

4. Given A, the area of a triangle, and the ratio of any side to the perpendicular on it from the opposite angle equal to $m : n$; find this side.

5. Find the area of a square inscribed in a circle whose radius is r.

6. Find the radius of a circle whose area is equal to that of three circles whose radii are r_1, r_2, r_3.

7. If d denote the difference between the radii of two circles, and S denote the difference between their areas, find the radius of each circle.

8. If A denote the area of a right-angled triangle, and h the hypothenuse, find the other two sides.

9. In a triangle whose sides are a, b, c, and whose area is S, we have given a, b, and S; it is required to find the remaining side c.

(1.) Interpret geometrically the values of c.
(2.) When will the two values of c be equal?

10. Find the ratio of the area of an equilateral triangle inscribed in a circle to that of the inscribed regular hexagon.

11. The sides of a triangle are 42, 40, and 37 feet; find the sides of a second triangle similar to the first, but four times larger in area.

12. Prove that the area of the regular hexagon inscribed in a circle is $\frac{3}{4}$ of the area of the circumscribing hexagon.

13. Indicate upon the faces of a pyramid the trace of a plane which divides the lateral surface into two parts in the proportion of 2 to 3.

14. To divide the triangle ABC, by a line DE parallel to AC, into two parts proportional to the numbers 8 and 9.

15. To divide the triangle ABC into *five* equal parts by lines D_1E_1, D_2E_2, D_3E_3, &c. parallel to AC.

16. To divide the triangle ABC into *five* parts proportional to the numbers 5, 6, 7, 8, 11, by lines D_1E_1, D_2E_2, &c. parallel to AC.

17. The total surface of a right circular cone is S, and its radius is R; find the length of the generating line.

18. The lateral surface of a cone is S, and the length of the generating line is G; find its volume.

19. If the slant height of a cone is equal to the radius of the base, prove that the lateral surface is equal to the surface of the base.

20. AB is the diameter of a semicircle whose centre is O; upon each of the radii, OA, OB, are described semicircles; find the volume generated by the surface between the three semicircles.

21. The interior radius of a circular tower is r, and the height is h; find the thickness of the wall when the volume of masonry is v.

22. The height of a cone is equal to its diameter; find the ratio of the surface of the base to the lateral surface.

23. The volume of the frustum of a cone = V, and the radii of the ends are R and r; find h.

24. From a sphere of radius r a segment is cut off, of which the convex surface is s; find the surface of its base.

25. Find the radius of a sphere whose volume is 2 cubic feet.

26. A cylinder circumscribes a sphere; find the ratios of the surface, and of the volume of the sphere to the total surface, and to the volume of the cylinder, r being the radius of the sphere.

27. An arc of a great circle of a° is l; find the volume of the sphere.

28. If a = the side of a regular pentadecagon inscribed in a circle whose radius is R, prove

$$a = \frac{R}{4}(\sqrt{10 + 2\sqrt{5}} + \sqrt{3} - \sqrt{15}).$$

29. Shew how to find the volume generated by a triangle turning round any line in its plane.

30. The radii of two circles are r_1 and r_2, the distance of their centres is a; find the length of their common chord. Find what relation must exist between the constants that the chord may be zero.

31. Find the sides of a right-angled triangle, knowing that they are three consecutive numbers.

32. If a = the side of a regular polygon inscribed in a circle whose radius is R, and A = the side of the similar circumscribed polygon, prove

$$(1.)\ A = \frac{2aR}{\sqrt{4R^2 - a^2}}.$$

$$(2.)\ a = \frac{2AR}{\sqrt{4R^2 + A^2}}.$$

33. In the fig. of *Euclid*, I. 1, if r denote the radius of each circle, find the area of the part common to both circles.

34. Three equal circles touch each other; find the area of the part between them, the radius of each circle being r.

35. Divide a circle of radius r into three equal parts by concentric circles.

36. Shew that the area of the regular inscribed octagon is equal to that of a rectangle whose adjacent sides are equal to the sides of the inscribed and circumscribed squares.

37. Find the ratio of the volumes of two cylinders whose lateral surfaces are equal.

38. Find the ratio of the lateral surfaces of two cylinders whose volumes are equal.

39. Shew that the areas of all triangles described about the same circle are proportional to their perimeters.

40. If α, β, γ denote the distances from the angular points of a triangle to the points of contact of the inscribed circle, shew that its radius is

$$\left(\frac{\alpha\beta\gamma}{\alpha+\beta+\gamma}\right)^{\frac{1}{2}}.$$

41. The perimeter of an equilateral triangle being 27 yards, find the area.

42. If the right angle were divided into *sixty* degrees instead of *ninety*, what would be the circular measure of $75°$?

43. An arc of $120°$ in one circle is equal to the whole circumference of another; compare their radii, and find how many degrees of the circumference of the smaller circle are equal in length to the radius of the greater.

44. The diameter of a circle is divided in the ratio of $1\frac{2}{3} : 2\frac{1}{2}$, and circles are described on the segments as diameters; compare the areas of the three circles.

45. Find the length of the arc which subtends an angle $6° 12' 36''$ in a circle whose radius is 100 yards.

46. How many cubic inches of iron will be required to form a garden roller, which is $\frac{1}{2}$ inch thick, with

an outer circumference of $5\frac{1}{2}$ feet, and a length of $3\frac{1}{2}$ feet.

47. If r be the radius of the inscribed circle of a triangle whose sides are a, b, c, and p_a, p_b, p_c the three perpendiculars, prove

$$p_a p_b p_c = \frac{(a+b+c)^3}{abc} r^3.$$

48. If a triangle, ABC, be formed by joining the centres of three circles which touch each other, and if s be half the sum of its sides, a, b, c, then the radii of the circles whose centres are A, B, C are respectively

$$s-a,\ s-b,\ s-c.$$

49. At the distance of 50 miles from a tower, its top just appeared in the horizon; find its height, having given the earth's diameter to be 7964 miles.

50. Find the area of a circle traced on a sheet of paper by a pair of compasses whose legs are 6 inches, and contain an angle of 120°; find also the area when the legs are at right angles to each other.

51. The hypothenuse, AB, of a right-angled triangle is trisected in D and E; prove that the sum of the squares of the sides of the triangle, $CDE = \frac{2}{3}AB^2$.

52. An equilateral triangle and a regular hexagon have the same perimeter; shew that the areas of their inscribed circles are as 4 : 3.

53. If p_a, p_b, p_c denote the lengths of the lines drawn from the angles of a triangle to the middle of the opposite sides, prove that side

$$c = \tfrac{2}{3}\sqrt{6(p_a^2+p_b^2)-3p_c^2}.$$

54. In a given circle is inscribed an equilateral triangle, and in that triangle another circle, and so on *ad infinitum;* shew that the sum of the circumferences of all the inner circles is equal to that of the given circle, and the sum of their areas to a third of its area.

55. Shew that the sum of the squares of the distances

of the centre of the inscribed circle from the angular points of a triangle

$$= ac + bc + ab - \frac{6abc}{a+b+c}.$$

56. If r, r_1, r_2, r_3 be the radii of the inscribed and escribed circles of a triangle, and R the radius of the circumscribing circle, prove

$$\text{Area of triangle} = \sqrt{rr_1r_2r_3},$$

and that $$r_1 + r_2 + r_3 - r = 4\text{R}.$$

57. Shew that the chord of a quadrant of a circle divides the circle into two parts in the ratio of 10 : 1 very nearly.

58. For 1 inch of rainfall, shew that the weight of water on an acre of ground is 100 tons nearly.

59. A circular hole is to be cut in a circular plate, so that the weight of the plate is reduced $\frac{1}{4}$; find the diameter of the hole.

60. The area of the regular hexagon inscribed in a circle is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

61. If a spherical shell, when formed into a solid sphere, be equal in volume to its own cavity, shew that the thickness of the shell = R × ·20629.

62. A cubic foot of copper is drawn into a wire $\frac{1}{10}$th of an inch in diameter; find its length.

63. A spherical shell weighs $\frac{19}{20}$ths of a solid sphere of the same size and material; find the diameter of the internal cavity.

64. A perfectly flexible rope of uniform thickness is closely coiled upon a level floor. If the diameter of the rope be $2a$, shew that the length of the first n complete coils will be

$$\pi an(2n+1);$$

shew also how to find the length of the $(n+1)$th coil.

ANSWERS TO EXERCISES.

Exercises (1).

1. (1.) 25. (2.) 19·209. (3.) 39·051.
   (4.) $17\sqrt{2}$. (5.) 809·84. (6.) 30·69.
2. (1.) 165·698. (2.) 10·82. (3.) 6·052.
3. (1.) 86·602. (2.) 1·24. (3.) 400.
4. 19·209. 5. 144. 6. 86·602.
7. $5\sqrt{2}$. 8. $50\sqrt{2}$. 9. $25\sqrt{2}$.
10. $2\sqrt{2}$. 11. $3\sqrt{2}$. 12. $12\sqrt{2}$.
13. $5\sqrt{3}$. 14. $5\sqrt{7}$. 15. 18.
16. 19·06. 17. 40 feet. 18. 4·49 feet.
19. 29·16 feet. 20. 8·66. 21. $\frac{2p}{\sqrt{3}}$.
22. $5(\sqrt{2}+1)$ 23. $5(\sqrt{2}-1)$. 24. $2(S+\sqrt{S^2-p^2})$.
25. (1.) $\frac{125}{2}\sqrt{7}$, $\frac{375}{4}\sqrt{7}$ (2.) $\frac{3}{10}\sqrt{231}$, $\frac{3}{5}\sqrt{231}$ (3.) $\frac{72}{19}\sqrt{15}$, $\frac{72}{17}\sqrt{15}$
   (4.) 2·4, 4 (5.) 4·8, 8 (6.) 4·84, 9·68
27. (1.) $\frac{800}{7}\sqrt{7}$. (2.) $\frac{400}{\sqrt{231}}$. (3.) $\frac{323}{4\sqrt{15}}$.
   (4.) 5. (5.) 10. (6.) $\frac{8}{3}\sqrt{15}$.

28. (1. $125\sqrt{3}$, 324) (2.) $25\sqrt{263}$, $50\sqrt{638}$)

29. (1.) 222. (2.) $10\sqrt{222}$. 30. 40·08.

31. (1.) $\frac{ab}{\sqrt{a^2+b^2}}$. (2.) $\frac{a^2}{\sqrt{a^2+b^2}}$, and $\frac{b^2}{\sqrt{a^2+b^2}}$.

Exercises (2).

1. 9·01. 2. 2 or 18. 3. 108.

4. $20\sqrt{3}$. 5. 10·43. 7. 31·81.

8. (1.) r. (2.) $\frac{2r}{3}$. (3.) $r\sqrt{3}$. (4.) $\frac{4r}{\sqrt{3}}$.

11. 5·19 : 5·65 : 6·08. 12. 5. 13. 6 inches.

14. (1.) $r\sqrt{2}$. (2.) $r\sqrt{5}$.

15. (1.) $\sqrt{D^2-(R\mp r)^2}$. (2.) $2\sqrt{Rr}$ or o.

16. $\frac{R}{2}(\sqrt{5}-1)$.

Exercises (3).

1. (1.) 125·6636. (2.) 21 ft. $5\frac{5}{7}$ in. (3.) $2828\frac{4}{7}$ yds.

2. (1.) $31\frac{9}{11}$ ft. (2.) 445 in. (3.) 3 ft. $11\frac{3}{11}$ in.
(4.) $153\frac{16}{242}$ ft. (5.) 17·5 ft. (6.) $\frac{1}{2\pi}$ in.

3. 4π inches. 4. (1.) $\frac{d}{2(\pi-1)}$. (2.) $4\frac{2}{3}$ feet.

5. (1.) $\frac{S}{2(\pi+1)}$. (2.) $2\frac{12}{29}$. 6. $157\frac{1}{7}$ feet.

7. 80 feet $1\frac{5}{7}$ in. 8. 480 times. 9. $1885\frac{5}{7}$ yds.

10. $22\frac{10}{11}$ in. 11. $21\frac{3}{14}$ in. 12. 4 : 63.

13. $4 : \pi : \pi\sqrt{2}$. 14. $\pi : 3 : 2\sqrt{3}$.

16. (1.) 137° 30′. (2.) 132° 30′. (3.) 20°.
(4.) 125°. (5.) 122° 30′. (6.) 15°.

17. At $8\frac{2}{11}$ minutes past 3 o'clock, and at $24\frac{6}{11}$ minutes past 3 o'clock.

18. (1.) 85° 56′ 37·5″. (2.) 114° 35′ 30″.
(3.) 68° 45′ 18″.

19. (1.) 42° 58′ 18″. (2.) 57° 17′ 45″.
(3.) 34° 22′ 39″

20. $8\frac{10}{11}$ ft. 21. $\frac{432}{5\pi}$. 22. 4° 46′ 8″. 23. 28° 38′ 52″.

24. (1.) $\left(\frac{m+n}{\pi r}\right)90°$. (2.) $\left(\frac{m-n}{\pi r}\right)90°$. 25. $114\frac{6}{11}$.

26. (1.) 38° 11′ 50″. (2.) 42° 58′ 18″. (3.) 57° 17′ 45″.
(4.) 95° 29′ 35″. (5.) 128° 54′ 56″. (6.) $\frac{\theta}{\pi r}(180°)$.

27. (1.) $\frac{\pi}{9}$. (2.) $\frac{\pi}{4}$. (3.) $\frac{3\pi}{5}$. (4.) $\frac{47\pi}{36}$. (5.) $\frac{5\pi}{8}$.
(6.) $\frac{5\pi}{6}$. (7.) $\frac{5\pi}{3}$. 28. $\frac{r_1}{r_2}=\frac{3}{2}$.

29. (1.) 65°. (2.) 52° 45′. (3.) 47° 34′ 45″.
(4.) 15°. (5.) 8° 50′. (6.) 82° 45′.

30. (1.) 69° 45′. (2.) 109°. (3.) 71° 35′.
(4.) 154° 50′. (5.) 120°. (6.) 45°.

31. (1.) $\frac{2\pi-3}{4}$. (2.) $\frac{4\pi-3}{4}$. 32. 45°.

Exercises (4).

1. (1.) 72. (2.) 4608. (3.) 146000.
(4.) 465750. (5.) 823440. (6.) 728000.
(7.) 101 feet 54 inches. (8.) 167 ft. 63 in.
(9.) $38\frac{1}{2}$ sq. yds.

2. (1.) 12100 square inches. (2.) 119025 square feet.
(3.) 314721 square yards. (4.) 7·58641 acres.
(5.) ·15625 acres. (6.) ·21025 acres.

3. (1.) $1125\sqrt{39}$. (2.) $8748\sqrt{17}$. (3.) $52900\sqrt{2}$.
(4.) 1453390·848. (5.) $1152\sqrt{5}$. (6.) $20000\sqrt{6}$.

4. (1.) 6050 sq. yds. (2.) 37812·5. (3.) 2708·48.
(4.) 618272. (5.) 1548300. (6.) 457·53125.

5. 45 feet. 6. $4\sqrt{7}$. 7. 50·16.

8. (1.) 21 square yards 5 feet 36 inches.
(2.) 213 square yards 7 feet 57 inches.
(3.) 67 square yards 32 inches.
(4.) 46 square yards 6 feet 36 inches.
(5.) 70 square yards 7 feet 76 inches.
(6.) 144 square yards 3 feet 90 inches.

9. 5 square yards 2 feet 76 inches.

10. 12 square feet 132 square inches.

11. 35 square yards 5 feet 48 inches.

12. $30\frac{1}{4}$ sq. yds. 13. 2 sq. yds. 6 ft. 14. 640 acres.

15. (1.) $44\frac{4}{9}$ yds. (2.) $74\frac{2}{27}$ yds. (3.) $133\frac{7}{18}$ yds.
(4.) $60\frac{18}{27}$ yds.

16. (1.) £12, 4s. 5d. (2.) £20, 7s. 4d.
(3.) £36, 13s. 7d. (4.) £16, 13s. 10d.

17. 3 ft. 4 in. 18. 6 ft. 19. $50907\frac{3}{7}$.

20. 440. 21. £618, 15s. 22. $S^2\{3-2\sqrt{2}\}$.

23. $D^2\{3+2\sqrt{2}\}$. 24. $\frac{D^2-d^2}{2}$.

25. (1.) 1000 sq. yds. (2.) 39 sq. ft. 27 in.

26. $60\frac{1}{2}$ yards. 27. $167\frac{1}{2}$ nearly.

28. (1.) $12\sqrt{42}$. (2.) 89·3. (3.) $256\sqrt{15}$.
(4.) $4\sqrt{3\cdot 19\cdot 211\cdot 323}$. (5.) $247500\sqrt{13}$.
(6.) $\frac{3}{10}\sqrt{2}$. (7.) 13·69. (8.) 58·2.
(9.) 155·25. (10.) $36\sqrt{3}$. (11.) $\cdot 000036\sqrt{3}$.
(12.) $\frac{3}{100}\sqrt{2}$.

29. (1.) 1 ac. 21 po. (2.) 8 ac. 1 po. (3.) 9 ac. 24 po.

30. $150\sqrt{14}$.

31. (1.) 15. (2.) $\frac{10}{3}\sqrt{105}$. (3.) 142.
33. (1.) 98·4375 : 225. (2.) 1 : 2.
36. 240 sq. in. 37. 1800 sq. yds. 38. 9680 sq. yds.
39. $\frac{5}{2}\sqrt{3}$. 42. 2 : $\sqrt[4]{27}$. 43. 9 : $4\sqrt{3}$.

Exercises (5).

1. $\frac{10}{3}\sqrt{2}$. 2. $\frac{50}{\sqrt{6}\sqrt{6}}, \frac{60}{\sqrt{6}\sqrt{6}}, \frac{70}{\sqrt{6}\sqrt{6}}$.
3. $66\frac{2}{3}$, 136, $197\frac{1}{3}$. 4. $2\sqrt{77}$ po., and $3\sqrt{77}$ po.
5. $14\sqrt{2}$, and $\frac{45}{2}\sqrt{2}$. 6. $(\frac{59}{17})^2 6\sqrt{33}$, and $(\frac{59}{7})^2 6\sqrt{33}$.
7. $\frac{13\times 78}{\sqrt{21}}, \frac{14\times 78}{\sqrt{21}}, \frac{15\times 78}{\sqrt{21}}$. 8. $\frac{10\sqrt{3}}{\pi}$. 9. 9 : 25.
10. $\sqrt{10} : \sqrt{7}$. 11. $54\frac{10}{21}$ sq. in. 12. 1 : 1214.
13. 1 : 11200. 14. $\frac{90}{121}a$ acres.

Exercises (6).

1. (1.) $5028\frac{4}{7}$ feet. (2.) 1320·785. (3.) 70714·28.
(4.) $2011\frac{3}{7}$ ac. (5.) π sq. in. (6.) $103\frac{51}{56}$ sq. ft.
2. (1.) $19886\frac{4}{11}$. (2.) $3181\frac{9}{11}$.
(3.) $\frac{1}{16\pi}$ of a square mile. (4.) 8 square feet nearly.
(5.) $\frac{\pi}{4}$. (6.) $286\frac{4}{11}$.
3. (1.) 3·07. (2.) $\frac{1}{\sqrt{\pi}}$ miles. (3.) 8·73 feet.
(4.) 27·7 yds. (5.) 12 po. nearly. (6.) 23·7 yds.
4. 154 square links. 5. 22·7. 6. 40·98.
7. $\frac{5}{2\pi}\sqrt[4]{3\pi^2}$. 8. $114\frac{2}{7}$. 9. $157\frac{8}{9}$.
10. $\frac{\text{Perimeter of square}}{\text{circumference of circle}} = \frac{2}{\sqrt{\pi}}$. 11. $1145\frac{5}{11}$.

12. $10\sqrt{10}$. 13. 436 tons. 14. $157\frac{1}{2}$. 15. $16\frac{2}{3}$.

16. 44π. 17. $10\times\frac{1}{\sqrt{3}}$, and $10\times\frac{\sqrt{2}}{\sqrt{3}}$. 18. $15\frac{1}{11}$.

19. $160\sqrt{35}$. 20. $\frac{24}{\sqrt{\pi}}$. 22. $36\frac{1}{7}$ and $\frac{23}{6}\pi$.

23. $3\sqrt{41}$ and 3.

Exercises (7).

1. (1.) $8\frac{1}{8}$ square feet. (2.) $127\frac{23}{56}$ square feet.
(3.) $64\frac{7}{360}$ square feet. (4.) $124\frac{17}{72}$. (5.) $176\frac{17}{24}$.
(6.) $135\frac{7}{8}$ square feet.

2. (1.) $9\frac{3}{8}$ square feet. (2.) $66\frac{2}{3}$ square feet.
(3.) $204\frac{1}{6}$. (4.) $234\frac{3}{8}$ square feet.
(5.) $155\frac{1}{24}$ square feet. (6.) $630\frac{3}{8}$.

3. 50 square feet. 4. 6 inches.

5. (1.) $3\frac{68}{72}$. (2.) $\frac{1}{36}(24+\sqrt{2})$. (3.) $\frac{5}{2}(3+5\sqrt{3})$.
(4.) $9\frac{1}{2}$ square feet nearly. (5.) $108(15+4\sqrt{6})$.
(6.) $1323+\frac{441}{2}\sqrt{3}$.

6. $\frac{288}{15}-\frac{5}{12}\sqrt{3}$. 7. $2\sqrt{3}\{5\pm\sqrt{25+24\sqrt{3}}\}$.

8. $67\frac{1}{2}$ square feet.

9. (1.) $\frac{\pi}{2}$ square feet. (2.) $\frac{81\pi}{4}$ square feet.
(3.) $\frac{27\pi}{4}$ square feet. (4.) $\frac{27\pi}{8}$ square feet.
(5.) $\frac{25\pi}{2}$ square feet. (6.) $\frac{539\pi}{2}$ square feet.

10. Height = radius. 11. $\left(\frac{144}{5\pi}-2\frac{1}{2}\right)$ inches.

12. $4\pm\frac{2}{\pi}\sqrt{4\pi^2+27\pi}$. 13. $\frac{3200}{\pi}$ square inches.

14. $\frac{10}{3}$ square feet. 15. $\frac{9}{2}\sqrt{\frac{6}{\pi}}$. 16. π sq. ft.

17. $\frac{\sqrt{10}}{2\sqrt{3\pi}}$ and $\frac{20}{3}$ sq. feet. 18. $\frac{3375\pi}{7}$ tons.

19. 40·32 square inches. 20. 39 square inches.

21. 7140 square inches. 22. 55π square feet.

23. $\frac{4}{3}$ square feet. 24. 126π square feet.

25. $103 \times 272{\cdot}2 \times \pi$ square inches.

26. $\frac{9\pi}{64}\sqrt{17}$ square yards, and $\frac{243\pi}{64}\sqrt{17}$ shillings.

27. $\frac{84\pi}{5}$ square inches.

28. 15π square inches, and 20π square inches.

30. $\frac{5053\pi}{72}$ square feet.

31. $\frac{252 \times 294{\cdot}35 \times \pi}{144}$ square feet.

32. $23\pi\sqrt{466}$ square inches.

33. $\frac{\text{Surface of cylinder}}{\text{surface of cone}} = \frac{2h}{(h^2+r^2)^{\frac{1}{2}}}$.

34. $\frac{52}{\pi}$ inches. 35. $\frac{90}{\pi}$ inches. 36. $\frac{5}{\pi}\sqrt{36\pi^2-25}$.

37. $\frac{200\pi}{27}\sqrt{13}$. 38. Slant height of cone cut off $= \frac{S}{\sqrt{2}}$.

39. (1.) 324π square inches. (2.) 900π square inches.
(3.) $\frac{400\pi}{9}$ square feet. (4.) $27{\cdot}04 \times \pi$ sq. feet.
(5.) π square yards. (6.) 49π square feet.

40. (1.) $\sqrt{\frac{30}{\pi}}$. (2.) $\frac{3}{2\sqrt{\pi}}$ feet. (3.) $3\sqrt{\frac{26}{\pi}}$ in.

41. $\frac{3}{2}\sqrt[4]{2}$. 42. surface of sphere = surface of cylinder.

43. 12π. 44. $\frac{2r}{3}$. 46. Height = diameter.

47. Height $= \frac{\text{diameter}}{n-2}$. 48. $\frac{5}{2(5+r)}$th part.

49. 64π square feet. 51. $\frac{25}{18\pi}$ inches.

52. If r = radius of earth in inches, then $4\pi r^2$ will be the number of square inches in the surface; therefore weight in tons $= \frac{3}{112}\pi r^2$.

53. $R = \sqrt{r^2 + r'^2}$. 54. $11\pi^2$ in. 55. $\pi^2 t(d-t)$ in.

56. $(C+c)\left\{\frac{1}{2\pi}(C-c)+h\right\}$. 57. 54π sq. in.

Exercises (8).

1. $2_{17}^{7}\frac{9}{28}$ cubic yards. 2. $17066\frac{2}{3}$ cubic feet.
3. (1.) 144 cubic feet 45 cubic inches.
   (2.) 3605 cubic feet 1 cubic inch.
   (3.) 1157 cubic feet 90 cubic inches.
4. (1.) $\sqrt[3]{110} = 4{\cdot}79$ inches.
   (2.) $2\sqrt[3]{307} = 13{\cdot}4$. (3.) 1·14 yards.
5. 8 inches, the other result being inadmissible.
6. $19\frac{1}{2}$ inches.
7. (1.) $51\frac{1}{4}$ cubic feet. (2.) $4\frac{19}{24}$ cubic yards.
8. (1.) $\frac{5}{3}$ feet. (2.) 24 feet.
9. (1.) $1234\frac{2}{7}$ square inches. (2.) square feet.
10. (1.) 153·3 gallons. (2.) 2917.
    (3.) 934 gallons.
11. 17 tons 4 cwts. $13\frac{2}{3}$ lbs.
12. $19\frac{18}{72}\frac{3}{00}\frac{7}{0}$ inches. 13. 515 gallons.
14. 6 tons nearly. 15. 12·4 tons.

Exercises (9).

1. (1.) $5\frac{1}{4}$ cubic feet. (2.) $31\frac{1}{2}$ cubic feet.

2. 9·596 cubic feet.

3. $\frac{8375}{2}\sqrt{3}$. 4. $\frac{720}{\sqrt{429}}$ inches.

5. (1.) $55\frac{11}{16}$ cubic feet. (2.) $16\frac{27}{32}$ cubic feet.
(3.) $982\frac{17}{42}$ cubic feet.

6. $\frac{25}{3\sqrt{30\pi}}$ feet. 7. $\frac{3125\pi}{1793}$ tons. 8. $9\frac{1}{11}$ inches.

9. $\frac{\text{Height of cone}}{\text{height of cylinder}} = \frac{3s^2\sqrt{3}}{2\pi r^2}$. 10. $10\frac{25}{36}$ cubic yds.

11. $3909\frac{9}{11}$. 12. 270π cubic inches.

13. $\frac{V}{V'} = \frac{h'}{h}$, or inversely as the heights.

14. $\frac{S}{S'} = \frac{r'}{r}$, or inversely as the radii.

Exercises (10).

1. $1125\sqrt{3}$. 2. $3200\sqrt{3}$. 3. $\frac{576000}{\sqrt{2}}$.

4. $600\frac{3}{49}$ cubic ft. 5. $4666\frac{20}{33}$ cubic feet.

6. $447\frac{1}{3}$ cubic ft. 7. 11 tons 6 cwts. $13\frac{2}{5}$ lbs.

8. $\frac{R}{r} = \frac{\sqrt{3}+1}{1}$. 9. $6\frac{16}{25}$ times.

11. 45 cubic in. 12. 320 cubic inches.

13. $V = \frac{\pi s^3}{6\sqrt{2}}$, $S = \frac{\pi s^2}{\sqrt{2}}$, $T = \frac{\pi s^2}{\sqrt{2}}\{\sqrt{2}+1\}$.

Exercises (11).

1. (1.) $\frac{1000\pi}{6}$ c. in. (2.) $\frac{125\pi}{6}$ c. ft. (3.) 36π c. ft.

2. (1.) $r = \frac{1}{2}\sqrt[3]{\frac{30000}{\pi}}$. (2.) $6\sqrt[3]{\frac{18}{\pi}}$.

3. (1.) $\frac{125}{6\sqrt{\pi}}$ cub. in. (2.) $\frac{1000}{6\sqrt{\pi}}$ cub. in.

4. $D = \sqrt[3]{6r^2h}$. 5. $R = \frac{1}{2}\sqrt[3]{2r^2h}$.

6. $\pi\{d^2t - 2dt^2 + \frac{4}{3}t^3\}$ cubic inches.

9. 16 : 9. 10. $\frac{\pi S^3}{4}$.

12. $V = 3\frac{a^2\sqrt{3}}{2} \times 2\pi\frac{a\sqrt{3}}{2}$ = surface of hexagon multiplied by circumference described by its centre.

14. h, the height, is found from the equation,

$$h^3 - \tfrac{3}{2}rh^2 + \frac{12m}{m+n}\cdot r^3 = 0.$$

15. $\frac{c^3}{6\pi^2}$. 17. $\frac{27}{113}$, and 1404928.

19. $\frac{(v^2+v'^2)^{\frac{1}{2}}}{(vv')^{\frac{1}{2}}}$ = diagonal. 20. $2R\sqrt{2}$.

22. $\frac{\pi b}{12}(4a^2 - b^2)$. 23. 1 : 2.

24. $R = \sqrt{\frac{V}{4\pi d} - \frac{d^2}{3}} + \frac{1}{2}d$, $r = \sqrt{\frac{V}{4\pi d} - \frac{d^2}{3}} - \frac{1}{2}d$.

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3. $AB = \dfrac{P^2}{\sqrt{P^2 - p_1^2}}$, $AC = \dfrac{P^2}{\sqrt{P^2 - p_2^2}}$,

$BC = P\left\{\dfrac{p_1}{\sqrt{P^2 - p_1^2}} + \dfrac{p_2}{\sqrt{P^2 - p_2^2}}\right\}$.

4. $\sqrt{\dfrac{2Am}{n}}$. 5. $2r^2$. 6. $\sqrt{r_1^2 + r_2^2 + r_3^2}$.

7. $\dfrac{S - d^2}{2\pi d}$, and $\dfrac{S + d^2(2\pi - 1)}{2\pi d}$.

8. $\frac{1}{2}\{\sqrt{p^2 + 4A} + \sqrt{p^2 - 4A}\}$,

and $\frac{1}{2}\{\sqrt{p^2 + 4A} - \sqrt{p^2 - 4A}\}$.

9. $c = \{a^2 + b^2 - 2\sqrt{a^2b^2 - 4s^2}\}^{\frac{1}{2}}$. 10. 1 : 2.

11. 84, 80, 74. 14. $BD = \dfrac{2\sqrt{2}}{\sqrt{17}}AB$.

15. $BD_1 = \dfrac{1}{\sqrt{5}}AB$, $BD_2 = \dfrac{\sqrt{2}}{\sqrt{5}}AB$, &c.

16. $BD_1 = \dfrac{\sqrt{5}}{\sqrt{37}}AB$, $BD_2 = \dfrac{\sqrt{11}}{\sqrt{37}}AB$, &c.

17. $\dfrac{S - \pi R^2}{\pi R}$. 18. $V = \dfrac{S^2}{3\pi^2 G^3}\sqrt{\pi^2 G^4 - S^2}$.

20. $\dfrac{\pi r^3}{3}$. 21. $t = \sqrt{\dfrac{V + \pi r^2 h}{\pi h}} - r$.

22. $\sqrt{5} : 1$. 23. $h = \dfrac{3V}{\pi(R^2 + r^2 + Rr)}$.

24. $S - \dfrac{S^2}{4\pi r^2}$. 25. $r = \sqrt[3]{\dfrac{3}{2\pi}}$.

26. $\dfrac{\text{Surface of sphere}}{\text{total surface of cylinder}} = \frac{2}{3} = \dfrac{\text{volume of sphere}}{\text{volume of cylinder}}$.

27. $\frac{1}{3}\left(\frac{2}{\pi}\right)^2 \cdot \left(\frac{180}{a}\right)^3 l^3$.

30. 8·96 inches.

31. 3, 4, 5.

33. $\frac{r^2}{6}(4\pi - 3\sqrt{3})$.

34. $\frac{r^2}{2}(2\sqrt{3} - \pi)$.

35. $r_1 = \frac{1}{\sqrt{3}}r$, $r_2 = \frac{\sqrt{2}}{\sqrt{3}}r$.

37. $\frac{v_1}{v_2} = \frac{r_1}{r_2} = \frac{h_2}{h_1}$.

38. $\frac{S_1}{S_2} = \frac{r_2}{r_1} = \frac{\sqrt{h_1}}{\sqrt{h_2}}$.

41. 70·1478 square yards.

42. $\frac{5}{8}\pi$.

43. $\frac{r_1}{r_2} = \frac{1}{3}$, $3\left(\frac{180^\circ}{\pi}\right) = 171^\circ\ 53'\ 14''$.

44. $4^\circ\ 9'\ 25''$.

45. $32\frac{1}{2}$ feet.

46. 2640.

49. 1657 feet.

50. (1.) $339\frac{3}{7}$. (2.) $226\frac{2}{7}$.

58. Diameter of hole = radius of plate.

61. 18327 feet.

62. $\frac{\text{Diameter of sphere}}{\sqrt[3]{20}}$.

THE END.

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